in the classroom

A rare example

A Surprising Fact about Triangles with a 60 degree Angle

Is the converse of a statement always true?

Ever posed this question to a class and then scanned your memory for good examples to clinch your argument? Here is one you could use.

$\mathcal{C} \otimes \mathcal{M}\alpha\mathcal{C}$

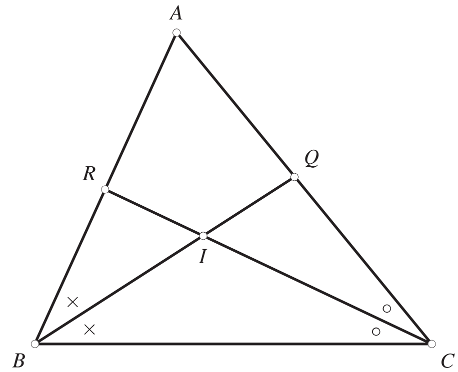
In the study of triangle geometry we get used to various pairs of theorems about isosceles triangles. Here are a few such pairs of statements, all with reference to a triangle ABC. Note their common element: the words '*and conversely*'.

(1) "If $AB = AC$ then $\angle B = \angle C$; and conversely." (That is, if $\angle B = \angle C$ then $AB = AC$.)

(2) "If $AB = AC$ then the medians from B and C have equal length; and conversely."

(3) "If $AB = AC$ then the altitudes from B and C have equal length; and conversely."

But occasionally we come across statements that go counter to this pattern; that is, the 'and conversely' fails. Here is one such. Given a $\triangle ABC$, let the internal bisectors of $\angle B$ and $\angle C$ meet each other at I (the incentre of the triangle), and let them meet the

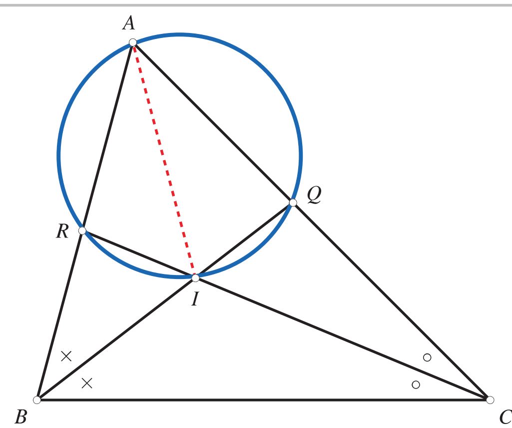


Suppose $AB = AC$. Then $\triangle BQC \cong \triangle CRB$ ('ASA' congruence); so $CQ = BR$.

Also, $\triangle BIR \cong \triangle CIQ$ ('SAS' congruence).

Hence $IQ = IR$.

Fig. 1



If points A, R, I, Q are concyclic, then $IQ = IR$.

Fig. 2

opposite sides (AC and AB) at Q and R respectively. For this configuration the following is true and easy to prove: *If $AB = AC$, then $IQ = IR$* (see Figure 1; the proof is given alongside).

Having seen so many statements about isosceles triangles of the form "If p then q" in which the propositions p and q can exchange places without any loss, we may now guess the following 'proposition': *If $IQ = IR$, then $AB = AC$*. But this turns out to be false!

What might a triangle look like in which $IQ = IR$ but $AB \neq AC$? To produce such a triangle we use a standard theorem in circle geometry: *Chords of a circle which subtend equal angles at a point on the circumference of the circle have equal length*.

Suppose that quadrilateral $ARIQ$ is cyclic (Figure 1). Since $\angle IAR = A/2 = \angle IAQ$, chords IR and IQ subtend equal angles at A; hence they have equal length. Therefore: *If points A, R, I, Q are concyclic, then $IQ = IR$.* (See Figure 2.)

Under what conditions will A, R, I, Q be concyclic? It is known that $\angle BIC = 90^\circ + A/2$. Hence $\angle QIR = 90^\circ + A/2$. Now a quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°. Hence $ARIQ$ is cyclic if and only if $A + 90^\circ + A/2 = 180^\circ$, which yields $A = 60^\circ$. So: *If $\angle A = 60^\circ$ then $IQ = IR$*. And this holds regardless of the relation between sides AB and AC! Hence from '$IQ = IR$' we *cannot* conclude that $AB = AC$. What we *can* conclude is this: *If $IQ = IR$, then either $AB = AC$, or $\angle A = 60^\circ$, or both.*

This may be rewritten as: *If $IQ = IR$, then either $AB = AC$, or $ARIQ$ is cyclic, or both.* We prove it in this form. We use the 'sine rule' which states that in any triangle, the ratio of the side to the sine of the opposite angle is the same for the three sides ($a/\sin A = b/\sin B = c/\sin C$). We call this common ratio the "side by sine ratio" of the triangle.

Examine $\triangle ARI$ and $\triangle AQI$. The side by sine ratio for $\triangle ARI$ is $IR/\sin A/2$, and for $\triangle AQI$ it is $IQ/\sin A/2$. These two ratios are equal, because $IQ = IR$. So the two triangles have the same side by sine ratio. This means in particular that $AI/\sin\angle AQI = AI/\sin\angle ARI$, and hence that $\sin\angle AQI = \sin\angle ARI$.

Two angles between 0° and 180° have equal sines just when they are equal or supplementary. Hence, either $\angle AQI = \angle ARI$, or $\angle AQI + \angle ARI = 180^\circ$. The first possibility holds when $AB = AC$, and the second possibility when $ARIQ$ is cyclic. Thus our claim is proved.

Acknowledgement

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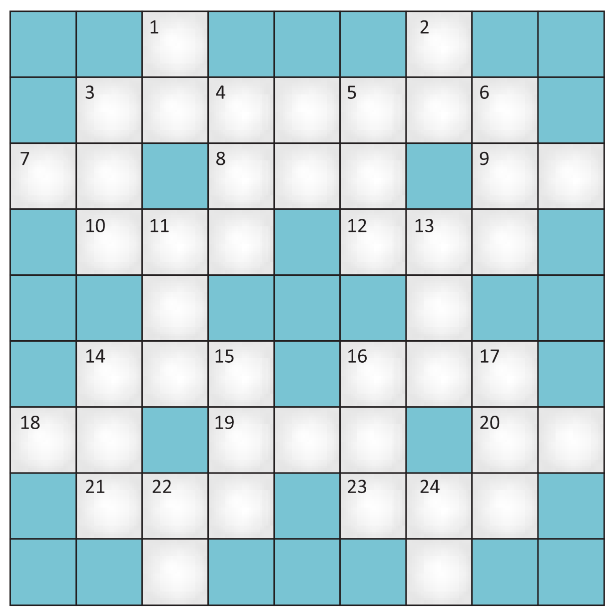
number crossword–2

Clues Across

3: 2D plus 38
5: 5D plus 5
7: Internal angle of an equilateral triangle
8: Add 10 to 2D times 11
9: Number of squares on a chess board
10: 24D times 3 with the digits mixed up
12: 7A+9A+18A+20A+1D
14: A Pythagorean triple: 313, 25, ------
16: 3D minus one digit of 7A
18: Last two digits of 15D plus 5
19: 10A minus 13D
20: Circumference of a circle with radius 1D/3
21: 5D jumbled up
23: Two thirds of 4D



Clues Down

1: Cube root of 9261
2: The largest perfect square below 100
3: An internal angle of a regular Pentagon
4: Largest three digit number
5: 14A minus 100
6: Largest number formed with 5, 6, 7
11: 1D in reverse times 9 plus 13
13: A full circle
14: 20A times 8 plus 10
15: 12A plus 6
16: 5A minus 1D
17: Three even digits in order
22: 24D divided by 3 plus 3
24: LCM of 3, 9 and 21