An 'Extreme Algebra' Question

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We came across this problem in a Facebook post [2] which in turn pointed to a YouTube video [3]: *Given that a, b, c are numbers such that* $a + b + c = 1$, $a^2 + b^2 + c^2 = 2$ *and* $a^3 + b^3 + c^3 = 3$, *find the value of* $a^5 + b^5 + c^5$.

In [3], the author describes this as an 'extreme algebra' problem, as the solution involves an 'extreme amount' of algebra. He solves the problem in a direct, straightforward manner, by multiplying the corresponding sides of the second and third equalities (thereby getting the expression $a^5 + b^5 + c^5$) and then somehow getting rid of the unwanted terms. (As we have already indicated, there is a substantial amount of algebra involved.)

We shall solve the problem by using an approach which draws on *the theory of equations*. This is an approach of great versatility and we highly recommend it to the reader.

A simpler example. We shall solve the following problem: *Given that a, b are numbers such that* $a + b = 1$ *and* $a^2 + b^2 = 2$, *find the value of* $a^5 + b^5$.

Consider the quadratic equation whose roots are a and b. Let this equation be

$$x^2 + ux + v = 0. \qquad (1)$$

The values of u and v may be found using the given data. Before doing so, let us see what we can glean from (1).

Multiplying through (1) by x^n, where n is any positive integer, we obtain:

$$x^{n+2} + ux^{n+1} + vx^n = 0. \qquad (2)$$

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Since a and b are solutions of (1), they must be solutions of (2) as well. Therefore we have:

$$\begin{cases} a^{n+2} + ua^{n+1} + va^n = 0, \\ b^{n+2} + ub^{n+1} + vb^n = 0. \end{cases} \tag{3}$$

Let $S_n = a^n + b^n$. (So $S_0 = 1 + 1 = 2$, $S_1 = a + b = 1$, $S_2 = a^2 + b^2 = 2$, and so on.) From (3), we obtain by addition, $S_{n+2} + uS_{n+1} + vS_n = 0$, i.e.,

$$\boxed{S_{n+2} = -uS_{n+1} - vS_n.} \tag{4}$$

The above relation, which shows how the members of the sequence $S_1, S_2, S_3, S_4, \ldots$ can be computed in terms of earlier members of the same sequence, is an example of a ***recurrence relation***. The study of such relations is of great importance in mathematics. See [1] for an introduction to this topic. A large number of such references can be found on the web.

Using the given data, $S_1 = 1$ and $S_2 = 2$, we may obtain as many terms as we wish of the sequence $S_1, S_2, S_3, S_4, \ldots$; we only need the values of u and v. To obtain these, we recall the theory of quadratic equations to deduce that

$$a + b = -u, \quad ab = v. \tag{5}$$

Since $a + b = 1$, we get $u = -1$; and since $a^2 + b^2 = 2$, we get

$$2ab = (a+b)^2 - \left(a^2 + b^2\right) = 1^2 - 2 = -1, \quad \therefore\ ab = -\frac{1}{2}, \quad \therefore\ v = -\frac{1}{2}.$$

It follows that

$$S_{n+2} = S_{n+1} + \frac{1}{2}S_n. \tag{6}$$

Using (6) repeatedly, we are able to compute successive terms of the sequence. We have displayed them in the table below.

n	0	1	2	3	4	5	6	7	8
S_n	2	1	2	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{3}{4}$	$6\frac{1}{2}$	$8\frac{7}{8}$	$12\frac{1}{8}$

We have described the method in detail, and we will now apply it to the given problem.

Back to the original problem. We return to the problem quoted at the start: *Given that a, b, c are numbers such that $a + b + c = 1$, $a^2 + b^2 + c^2 = 2$ and $a^3 + b^3 + c^3 = 3$, find the value of $a^5 + b^5 + c^5$.*

Let a, b, c be the roots of the following cubic equation:

$$x^3 + ux^2 + vx + w = 0. \tag{7}$$

By the factor theorem, the following identity must hold:

$$x^3 + ux^2 + vx + w = (x-a)(x-b)(x-c), \tag{8}$$

implying the following relations between $\{u, v, w\}$ and $\{a, b, c\}$:

$$\begin{cases} u &= -(a+b+c), \\ v &= ab + bc + ca, \\ w &= -abc. \end{cases} \tag{9}$$

Let $P_n = a^n + b^n + c^n$. Since $P_1 = 1$ (given), it follows that $u = -1$. Since $P_2 = 2$ (given), it follows from the identity

$$2(ab + bc + ca) = (a+b+c)^2 - \left(a^2 + b^2 + c^2\right) \tag{10}$$

that $2v = 1^2 - 2 = -1$, and so $v = -1/2$.

We now set up a recurrence relation for the sequence $P_1, P_2, P_3, \ldots$ just as we did earlier. Thus we have, from (7),

$$x^3 = -ux^2 - vx - w = x^2 + \frac{x}{2} - w.$$

Therefore, for all positive integers n,

$$x^{n+3} = x^{n+2} + \frac{x^{n+1}}{2} - wx^n. \tag{11}$$

Each of a, b, c must satisfy (11). Hence, summing over a, b, c, we obtain

$$P_{n+3} = P_{n+2} + \frac{P_{n+1}}{2} - wP_n. \tag{12}$$

We know that $P_3 = 3$ (given). Trivially, it is true that $P_0 = a^0 + b^0 + c^0 = 3$. Substituting in (12) with $n = 0$, we get

$$3 = 2 + \frac{1}{2} - 3w, \qquad \therefore\ w = -\frac{1}{6}.$$

It follows that:

$$\boxed{P_{n+3} = P_{n+2} + \frac{P_{n+1}}{2} + \frac{P_n}{6}.} \tag{13}$$

We are now in a position to compute as many members of the sequence as we wish. Using the recurrence relation (13) repeatedly, we obtain the table displayed below.

n	1	2	3	4	5	6	7	8	9	$\cdots$
P_n	1	2	3	$4\frac{1}{6}$	6	$8\frac{7}{12}$	$12\frac{5}{18}$	$17\frac{41}{72}$	$25\frac{5}{36}$	$\cdots$

This yields the required answer, $P_5 = 6$, but as may be observed, we have obtained much more.

References

1. Tutorials point, Discrete Mathematics – Recurrence Relation, https://www.tutorialspoint.com/discrete_mathematics/discrete_mathematics_recurrence_relation.htm
2. Valdosta maths club, https://www.facebook.com/groups/valdostamathclub/permalink/855020178186192/
3. Extreme algebra question, https://www.youtube.com/watch?v=1TBVeuOcY1w&fbclid=IwAR0ceGeNmSjm1vE_Y-iHz8hNNNtSmZJoZnpbKmCBrSIRv0wD9xfCPaW4nLE

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