Trigonometric Proof of Ptolemy's Theorem

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In this short note, we provide a trigonometric proof of Ptolemy's theorem.

Theorem (Ptolemy)**.** If quadrilateral $ABCD$ is cyclic then $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

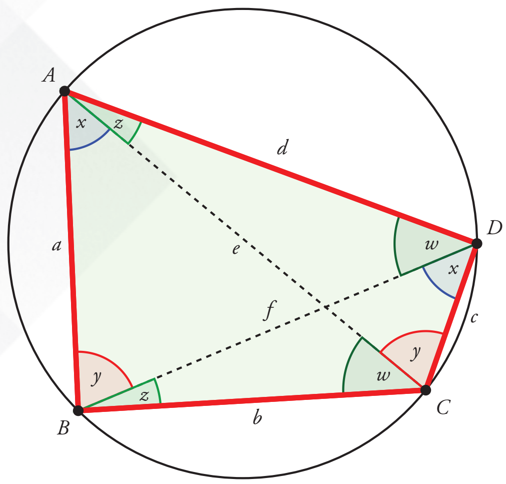


Figure 1.

Proof. Let

$$\measuredangle CAB = \measuredangle CDB = x, \qquad \measuredangle ABD = \measuredangle ACD = y,$$
$$\measuredangle DAC = \measuredangle DBC = z, \qquad \measuredangle ACB = \measuredangle ADB = w.$$

Also let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, $BD = f$.

Keywords: Ptolemy's theorem, quadrilateral, cyclic, sine rule

By the sine rule, since quadrilateral *ABCD* is cyclic, we have

$$\begin{aligned}
a/\sin w &= d/\sin y = f/\sin(y+w),\\
b/\sin x &= a/\sin w = e/\sin(x+w),\\
c/\sin z &= b/\sin x = f/\sin(x+z),\\
d/\sin y &= c/\sin z = e/\sin(y+z).
\end{aligned}$$

From the second and third equalities we get:

$$\frac{ac}{\sin w\cdot\sin z}=\frac{ef}{\sin(x+w)\cdot\sin(x+z)},$$

and from the third and fourth equalities we get:

$$\begin{aligned}
\frac{bd}{\sin x\cdot\sin y}&=\frac{ef}{\sin(x+z)\cdot\sin(y+z)}\\
&=\frac{ef}{\sin(x+z)\cdot\sin(x+w)},
\end{aligned}$$

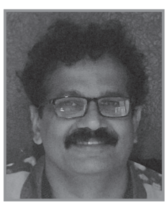
since $x+y+z+w=\pi$ and $\sin(\pi-u)=\sin u$ for any angle u. Therefore:

$$\begin{aligned}
\frac{ac}{ef}&=\frac{\sin z\cdot\sin w}{\sin(x+w)\cdot\sin(x+z)},\\
\frac{bd}{ef}&=\frac{\sin x\cdot\sin y}{\sin(x+z)\cdot\sin(x+w)}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{ac+bd}{ef}&=\frac{\sin z\cdot\sin w+\sin x\cdot\sin y}{\sin(x+w)\cdot\sin(x+z)}\\
&=\frac{\cos(z-w)-\cos(z+w)+\cos(x-y)-\cos(x+y)}{\cos(z-w)-\cos(2x+z+w)}\\
&=\frac{\cos(z-w)+\cos(x+y)+\cos(x-y)-\cos(x+y)}{\cos(z-w)+\cos(x-y)}\\
&=1 \qquad \text{(repeatedly using } x+y+z+w=\pi).
\end{aligned}$$

Therefore $ac+bd=ef$, or $AB\cdot CD+BC\cdot AD=AC\cdot BD$. □



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