Adventure with Quadratic Equations

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Quadratic equations form an important topic in most school curricula and students are exposed to problems of the following types:

(a) Finding the roots;

(b) Determining the nature of roots without explicitly finding them;

(c) Determining the range of values of a quadratic expression or a rational function having a quadratic numerator or denominator;

(d) Word problems where the quadratic equation has to be formulated and then solved.

More often than not the problems on quadratic equations discussed in the school curricula can be solved by a routine application of a handful of formulae and laborious algebraic manipulations. As a result the topic may appear to be charmless to the students and teachers alike. The goal of this article is to debunk this notion by way of illustrative examples that will bring the adventurous side of quadratic equations to the fore.

The first example is a word problem formulated as a simple game being played by two friends, Akbar and Birbal.

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Example 1. Akbar writes down the quadratic equation

$$ax^2 + bx + c = 0$$

with positive integer coefficients a, b, c. Then Birbal changes one, two or all three '$+$' signs to '$-$'. Akbar wins if both roots of the (modified) equation are integers. Otherwise (if there are no real roots or at least one of them is not an integer), Birbal wins. Can Akbar choose the coefficients in such a way that he will always win?

Solution. How do we start? There are eight possible equations (including the one written by Akbar) which can be divided into two groups, P and N, according to whether the sign of a is "$+$" or "$-$"; each group has four equations. Note that every equation in N may be obtained by multiplying by -1 an equation in P, and distinct equations in N are obtained from distinct equations in P. Thus it suffices to deal with the equations in any one of the two groups. We choose that group to be P. So, Akbar has to choose the coefficients a, b, c in such a way that all the four equations

(i) $ax^2 + bx + c = 0$

(ii) $ax^2 + bx - c = 0$

(iii) $ax^2 - bx + c = 0$

(iv) $ax^2 - bx - c = 0$

have integer roots. Since the sum of the roots and their product for the equations listed above are $\pm b/a$ and $\pm c/a$, to ensure that the roots are integers, it is wise for Akbar to choose $a = 1$. Observe that the roots of equations (iii) and (iv) differ from those of (i) and (ii), respectively, by a factor of -1. Thus, ensuring that the following equations have integer roots,

$$x^2 + bx + c = 0,$$
$$x^2 + bx - c = 0$$

is enough for Akbar to win the game.

Playing with small integers, Akbar will sooner or later obtain the winning quadratics

$$x^2 + 5x + 6 = 0 \quad \text{and} \quad x^2 + 5x - 6 = 0.$$

If Akbar decides to continue his search with vigour and enthusiasm, he will obtain several possible choices for the pair (b, c) that will guarantee his win against Birbal. To list a few:

$$(b, c) = (13, 30),\ (17, 60),\ (25, 84),\ \ldots.$$

But are there finitely many such pairs or infinitely many? This question is answered below. $\square$

Proposition. *Let the positive integers b and c be such that both the quadratic equations*

$$x^2 + bx + c = 0, \quad x^2 + bx - c = 0$$

have integer roots. Then there exists a right-angled triangle with hypotenuse b and area c.

Proof of proposition. As both quadratics have integer roots, both $b^2 - 4c$ and $b^2 + 4c$ are perfect squares. Let

$$x^2 = b^2 - 4c, \quad y^2 = b^2 + 4c,$$

where x, y are positive integers. Then

$$b^2 = \frac{y^2 + x^2}{2} = \left(\frac{y + x}{2}\right)^2 + \left(\frac{y - x}{2}\right)^2$$

and

$$c = \frac{1}{2}\left(\frac{y+x}{2}\right)\left(\frac{y-x}{2}\right).$$

Observe that both x^2 and y^2 have the same parity because their difference is even, and $y > x$. Thus x and y have the same parity, hence $(y+x)/2$ and $(y-x)/2$ are positive integers, and indeed b is the hypotenuse of the right-angled triangle with sides

$$\frac{y-x}{2}, \quad \frac{y+x}{2}, \quad \sqrt{\frac{y^2+x^2}{2}}$$

whose area is c. (The reader may verify that the three numbers are the sides of a non-degenerate triangle.)

So Akbar can choose any Pythagorean triple (r, s, t) and set $b = t$, $c = \frac{1}{2}rs$ to ensure his triumph over Birbal. Since there are infinitely many Pythagorean triples, Akbar has infinitely many pairs (b, c) at his disposal. $\square$

The next example involves permutation of the coefficients of a quadratic trinomial.

Example 2. Three nonzero real numbers a, b and c are given. We are told that if they are written in any order as the coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root?

Solution. Since each quadratic trinomial has real coefficients, if each has a real root, then each must have two real roots. Suppose there is one trinomial which does not have a positive root. Without loss of generality, let it be $ax^2 + bx + c$. Since $c \neq 0$, we know that 0 is not a root of this trinomial. Let $-u$ and $-v$ be its roots where $u > 0$ and $v > 0$. Then

$$ax^2 + bx + c = a(x+u)(x+v),$$

and we notice that the signs of $b = a(u+v)$ and $c = auv$ are the same as that of a. Therefore, without loss of generality, we may assume that a, b, c are positive.

But according to the problem, each of $ax^2 + bx + c$, $bx^2 + cx + a$ and $cx^2 + ax + b$ has two real roots. Therefore

$$b^2 \geq 4ac, \quad c^2 \geq 4ab, \quad a^2 \geq 4bc.$$

These inequalities lead to

$$(abc)^2 \geq 64(abc)^2,$$

an absurd result unless $abc = 0$, which is impossible. Thus each of the six quadratic trinomials has a positive root. $\square$

Here is a teaser for the reader.

Problem. Let a, b, c be three integers in arithmetic progression. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are integers, find the ratio $a : b : c$.

Using the Intermediate Value Theorem. Sometimes it is easier to establish the existence of a real root of a quadratic trinomial by exhibiting two real numbers at which it has opposite sign, rather than by explicitly computing the discriminant and showing that it is non-negative. The existence of a real root then follows from the Intermediate Value Theorem for a continuous function.The following example highlights this fact.

Example 3. Suppose P_1, P_2, P_3 are quadratic trinomials with positive leading coefficients and real zeros. Show that if each pair of them has a common zero, then the trinomial $P_1 + P_2 + P_3$ also has real zeros.

Solution. Let the common zero of P_1 and P_2 be β, and that of P_2 and P_3 be γ. Then the zeros of P_2 are β and γ, and if α is the other zero of P_1, then the zeros of P_3 are γ and α. Without loss of generality we may assume that $\alpha \le \beta \le \gamma$. So $P = P_1 + P_2 + P_3$ can be written as

$$P(x) = a_1(x-\alpha)(x-\beta) + a_2(x-\beta)(x-\gamma) + a_3(x-\gamma)(x-\alpha), \tag{1}$$

where $a_i > 0$ for $i = 1, 2, 3$. Then

$$P(\alpha) = a_2(\alpha-\beta)(\alpha-\gamma), \qquad P(\beta) = a_3(\beta-\gamma)(\beta-\alpha), \qquad P(\gamma) = a_1(\gamma-\alpha)(\gamma-\beta).$$

Observe that $P(\alpha) \ge 0 \ge P(\beta)$ and $P(\beta) \le 0 \le P(\gamma)$. This shows that P has a real zero between α and β, and another real zero between β and γ. □

The following example is from the Belarusian Mathematical Olympiad and it illustrates the level of challenge that a problem on quadratic equation can have.

Example 4. We call two quadratic trinomials $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + cx + d$ **friendly** if each of them has distinct real roots, and if $x_1 < x_2$ are the roots of $P(x)$ and $x_3 < x_4$ are the roots of $Q(x)$, then $x_1 + x_3$ and $x_2 + x_4$ are the roots of the quadratic trinomial $x^2 + (a+c)x + (b+d)$. Let M be the set of pairwise friendly trinomials consisting of at least three trinomials. Prove that 0 is a root of every trinomial from the set M.

Solution. Let $P(x) = x^2 + ax + b$, $Q(x) = x^2 + cx + d$ and $R(x) = x^2 + ex + f$ be three pairwise friendly trinomials in M. Suppose $x_1 < x_2$ are the roots of $P(x)$, $x_3 < x_4$ are the roots of $Q(x)$, and $x_5 < x_6$ are the roots of $R(x)$. Then $x_1 + x_3$, $x_2 + x_4$ are the roots of

$$P(x) + Q(x) - x^2 = x^2 + (a+c)x + (b+d),$$

$x_3 + x_5$, $x_4 + x_6$ are the roots of

$$Q(x) + R(x) - x^2 = x^2 + (c+e)x + (d+f),$$

and $x_5 + x_1$, $x_6 + x_2$ are the roots of

$$R(x) + P(x) - x^2 = x^2 + (e+a)x + (f+b).$$

Observe that

$$b + d = x_1x_2 + x_3x_4 = (x_1 + x_3)(x_2 + x_4),$$

whence

$$x_1x_4 + x_2x_3 = 0.$$

Similarly from

$$d + f = x_3x_4 + x_5x_6 = (x_3 + x_5)(x_4 + x_6)$$

and

$$f + b = x_5x_6 + x_1x_2 = (x_5 + x_1)(x_6 + x_2)$$

we obtain

$$x_3x_6 + x_4x_5 = 0$$

and

$$x_1x_6 + x_2x_5 = 0.$$

From the above we obtain:

$$x_1x_4 = -x_2x_3, \qquad x_3x_6 = -x_4x_5, \qquad x_2x_5 = -x_1x_6,$$

which lead to

$$(x_1x_4)(x_3x_6)(x_2x_5) = (-x_2x_3)(-x_4x_5)(-x_1x_6),$$

or:

$$x_1x_2x_3x_4x_5x_6 = 0.$$

Therefore at least one of $x_1, x_2, x_3, x_4, x_5, x_6$ is zero. Moreover, since the roots of each of $P(x)$, $Q(x)$ and $R(x)$ are distinct, at most one root of each can be zero. Suppose $x_1 = 0$. Then $x_2 \neq 0$ and it follows that $x_3 = 0$. Therefore $x_4 \neq 0$. From the above we get $x_5 = 0$ and hence $x_6 \neq 0$. This shows that 0 is a root of each of $P(x)$, $Q(x)$, and $R(x)$. The reader may verify that we would have reached the same conclusion if we had assumed that $x_k = 0$ for $k = 2, 3, 4, 5, 6$ instead of $x_1 = 0$.

The next example is from the Russian Mathematical Olympiad.

Example 5. A quadratic polynomial $f(x) = ax^2 + bx + c$ has no real roots. It is given that b is a rational number, and exactly one of c and $f(c)$ is a rational number. Is it possible for the discriminant of $f(x)$ to be a rational number?

Solution. Suppose c is a rational number. Then, by hypothesis, $f(c) = c\,(ac + b + 1)$ is irrational. Since b and c are rational, a must be irrational. Therefore the discriminant $D = b^2 - 4ac$ is irrational.

Suppose $f(c)$ is rational but c is irrational. Note $f(c) \neq 0$, since f does not have any real root. Then $(ac + b + 1) \neq 0$ and is irrational. But b is rational. Therefore ac is irrational and hence $D = b^2 - 4ac$ is irrational.

Our last example is a problem with a simple statement which can be generalised to a much deeper result.

Example 6. Let a and b be positive integers such that $n^2 + 2an + b$ is a perfect square for all integers n. Then the quadratic trinomial $x^2 + 2ax + b$ is the square of a linear polynomial.

Solution. Let $f(x) = x^2 + 2ax + b$. Then $f(-a) = b - a^2$ is a perfect square. Let c be such that $b - a^2 = c^2$. If $c = 0$ we are done. We will prove that under the given hypothesis c cannot be nonzero. Suppose $c \neq 0$. Then

$$f(c - a) = c^2 + b - a^2 = 2c^2.$$

But $2c^2$ cannot be a perfect square unless $c = 0$, as the exponent of 2 in $2c^2$ is odd if $c \neq 0$. Therefore $c = 0$, implying $b = a^2$ and

$$f(x) = (x + a)^2.$$

The general statement illustrated by the above example is the following:

Proposition. *If $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is a k^{th} power for some positive integer k, then there exists a polynomial $Q(x)$ with integer coefficients such that $P(x) = (Q(x))^k$.*

We hope that the reader will find this escapade involving quadratics stimulating enough to plunge into another on his or her own.

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