The Rascal Triangle

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This article is written with two purposes in mind, to illustrate two important themes. The first is to highlight the fact that even the most familiar settings can throw up surprises. The second is to point out that in a sufficiently nurturing classroom atmosphere, where the teacher and student are willing to listen to each other and travel together along uncharted territory, a great deal of learning can happen and students can go very far indeed in their explorations.

In a classroom of a school in America, a mathematics teacher had assigned the following work to his students. He had displayed the first four lines of the well-known Pascal triangle (namely, rows 0, 1, 2 and 3), and had asked them to guess or to deduce what could be the next few lines. What he had displayed was the array shown in Table 1 (see [2, 3]).

Presumably, he wanted them to spot the defining Pascal property and to therefore propose that the next row should be: $1, 4, 6, 4, 1$; the one after that: $1, 5, 10, 10, 5, 1$; and so on.

Instead, the students surprised him by proposing that the row after $1, 3, 3, 1$ should be $1, 4, 5, 4, 1$ and the one after that should be $1, 5, 7, 7, 5, 1$. This seemed wrong, and the teacher challenged them.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	?		?		?		?		?	
?		?		?		?		?		?

Table 1. The familiar Pascal array

Then he discovered that the students had used a totally different rule to generate the rows! In the Pascal triangle each new row is generated additively, using the numbers in the row above it (namely, by adding the two numbers closest to the entry to be filled). Thus if we have:

a		b
	x	

then the new entry x is given by: $x = a + b$. But in the rule used by the students, the numbers in each new row are computed using the *two* rows preceding it. Thus if we have:

	a	
b		c
	x	

then the new entry x is given by:

$$x = \frac{bc + 1}{a}.$$

It may be appropriate to call the generating rule for the Pascal array a "triangular rule" (based on the underlying shape), and the one used by the students a "diamond rule."

The diamond rule used by the students looks more complicated than the triangular Pascal rule: it involves both multiplication and division, whereas in the Pascal triangle we only do addition. In the Pascal triangle, all numbers will naturally be positive integers. But in the new array being considered, it is not at all obvious what kind of numbers are going to be produced. One wonders: do we get non-integral numbers? The surprising discovery we make is: despite the division, all the numbers do turn out to be positive integers. And though in one sense the new array is more complicated than the Pascal array, as it involves a more complicated recurrence rule, in another sense it turns out to be much simpler, as we shall see below.

The students who put forward this new rule and explored this new array whimsically named it the *Rascal triangle*. They also referred to the generating rules in a non-standard manner, using the cardinal directions. Thus, they referred to the rule that generates the Pascal array as the 'East-West rule':

WEST		EAST
	x	

with $x = \text{WEST} + \text{EAST}$; and they described their own rule in the following way [1]. If the configuration is:

	NORTH	
WEST		EAST
	x	

1									
1	1								
1	2	1							
1	3	3	1						
1	4	5	4	1					
1	5	7	7	5	1				
1	6	9	10	9	6	1			
1	7	11	13	13	11	7	1		
1	8	13	16	17	16	13	8	1	
1	9	15	19	21	21	19	15	9	1

Table 2. The first ten rows of the Rascal array

then x is given by:

$$x = \frac{\text{WEST} \times \text{EAST} + 1}{\text{NORTH}}.$$

The rest of this brief article will be devoted to finding a generating formula for the Rascal triangle.

Uncovering a formula for the array

Our first task will be to prove that all the numbers in the Rascal array are positive integers. To our surprise, we find that the simplest way to prove this is by establishing more: we discover and prove a generating formula for the array, and this formula then directly shows that all the entries are positive integers. This is an illustration of the maxim that in mathematics, "less can be more and more can be less" (meaning that it can be simpler to prove more than what is required). In this particular case, what we shall do is to use the diamond generating rule repeatedly to generate as much of the array as we can; then we shall study the array and guess its patterns and thus deduce its generating formula. Table 2 shows the first ten rows of the Rascal array.

Look at the diagonals of this array; what a delightful surprise! We see a collection of arithmetic progressions (see Table 3).

In indexing the diagonals, we intentionally start the numbering from 0 rather than 1. So Term(0) of each diagonal is 1. We now readily guess the generating formula for these diagonals: Term(k) of Diagonal(n) is

Terms	#0	#1	#2	#3	#4	#5	#6	...
Diagonal 0	1	1	1	1	1	1	1	...
Diagonal 1	1	2	3	4	5	6	7	...
Diagonal 2	1	3	5	7	9	11	13	...
Diagonal 3	1	4	7	10	13	16	19	...
Diagonal 4	1	5	9	13	17	21	25	...
Diagonal 5	1	6	11	16	21	26	31	...
Diagonal 6	1	7	13	19	25	31	37	...
Diagonal 7	1	8	15	22	29	36	43	...

Table 3. First few diagonals of the Rascal array

1									
1	1								
1	2	1							
1	3	3	1						
1	4	5	4	1					
1	5	7	7	5	1				
1	6	9	10	9	6	1			
1	7	11	13	13	11	7	1		
1	8	13	16	17	16	13	8	1	
1	9	15	19	21	21	19	15	9	1

Table 4

$kn+1$. *Note that this is purely an educated guess, based on the observed pattern.* We have not yet actually proved the formula.

We number the rows the same way, starting from 0. Now observe that as we traverse Row(n), Entry(0) lies on Diagonal(n) (remember that we are traversing the diagonals from top left to bottom right), the next entry lies on Diagonal$(n-1)$, the entry after that lies on Diagonal$(n-2)$, and so on. So Entry(k) of Row(k) is Entry(k) of Diagonal$(n-k)$. The array shown in Table 4, with Row(5) and Diagonal(2) highlighted, may make the meaning of this statement clearer: Entry(3) on the row (namely, 7) is also Entry(3) on the diagonal.

As per the conjectured formula, Entry(k) of Diagonal$(n-k)$ should be $(n-k)k+1$. Hence we have the following:

Conjecture. *Entry(k) in Row(n) of the Rascal array is* $k(n-k)+1 = kn-(k^2-1)$.

So according to this conjecture, the numbers in Row(n) are the following:

$$1,\ n,\ 2n-3,\ 3n-8,\ 4n-15,\ 5n-24,\ \ldots.$$

Having discovered a formula that seems to fit, it is an easy matter to verify that it is correct, using mathematical induction. First we verify that it is true for the first few rows of the Rascal triangle; this is easily done (please do so for yourself). Next, we observe that the diamond rule connects the following four elements:

	Entry $(n-1,k)$	
Entry (n,k)		Entry $(n,k+1)$
	Entry $(n+1,k+1)$	

As per the conjectured formula, where Entry $(n,k)=k(n-k)+1$, this translates to:

	$k(n-1)-k^2+1$	
$kn-k^2+1$		$k(n-2)-k^2+n$
	$k(n-1)-k^2+n+1$	

Hence we must verify whether the following is true:

$$\frac{(kn - k^2 + 1) \times [k(n-2) - k^2 + n] + 1}{k(n-1) - k^2 + 1} = k(n-1) - k^2 + n + 1.$$

We leave it to you to verify that equality does indeed hold here. Hence the conjectured formula is true.

Having found the generating formula of the Rascal array, we see that this array is indeed simpler than the Pascal array, despite the fact that its generating rule looks more complicated. The patterns in the diagonals of the Rascal array are certainly far simpler than those in the Pascal array; they are arithmetic progressions (and APs are surely the simplest kind of progression). And the general term in the array is a two-variable quadratic expression. The corresponding formula for the Pascal array involves the factorial numbers, which are considerably more complicated than the square numbers.

Closing remarks. It is worth noting the conditions that enabled such a phenomenon to occur, namely: the discovery of a new array of numbers in a classroom setting. It needed a teacher who was not stuck to the 'book answer' and instead was alert to the possibility of a fresh discovery within the boundaries of a classroom; a teacher who was willing to travel with the students and go that extra mile to understand the thinking of his students. It also required that this entire event be recorded. Teachers must be encouraged to document and share accounts of such events.

References

1. Stover, Christopher. "Rascal Triangle." From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/RascalTriangle.html
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3. "The Rascal Triangle." http://www.maa.org/publications/periodicals/college-mathematics-journal/rascal-triangle

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