Lessons from Proofs both False and True

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Typically, it is only after a false proof has reached some absurd conclusion that one backtracks to see what went wrong. Often one learns something of interest. We wondered if we routinely miss such lessons by not analysing 'correct' proofs just as diligently. We decided to investigate.

We first summarise a popular false proof [1] showing that all triangles are isosceles. We then analyse what we would have questioned in the resolution of this false proof had it too given an absurd result. Finally, we develop our own counter to the false proof.

The False Proof: All Triangles are Isosceles

With reference to Figure 1, which shows an arbitrary $\triangle ABC$:

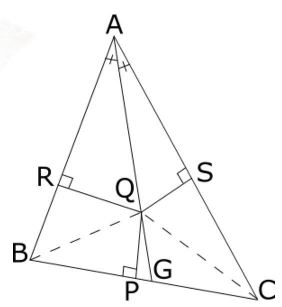

Figure 1.

Step 1. Let the angle bisector AG of $\measuredangle A$ and perpendicular bisector PQ of BC intersect at Q.

Step 2. Draw QR and QS perpendicular respectively to AB and AC.

Step 3. By Angle-Angle-Side congruence (AAS), $\triangle RAQ \cong \triangle SAQ$ so $AR = AS$. Also, $RQ = SQ$.

Step 4. Since PQ is the perpendicular bisector of BC, by Side-Angle-Side congruence (SAS), $\triangle BPQ \cong \triangle CPQ$. So $QB = QC$.

Step 5. From Steps 3 and 4, by Right-Hypotenuse-Side congruence (RHS), $\triangle RQB \cong \triangle SQC$ so $RB = SC$.

Step 6. Thus, $AR + RB = AS + SC$, i.e., $AB = AC$, i.e., $\triangle ABC$ is isosceles.

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If Q lies outside the triangle as shown in Figure 2, the proof is identical till Step 5.

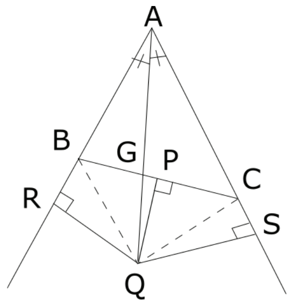

Figure 2.

Then, we say: $AR - RB = AS - SC$, thus, $AB = AC$.

The Resolution

The false proof misleads us in two places [4]:

Flaw #1: The point of intersection Q cannot lie inside the triangle as suggested by Figure 1.

But the proof works for Figure 2 (with Q outside the triangle) as well!

Flaw #2: In Figure 2, the perpendiculars QR and QS cannot both meet AB and AC respectively outside the triangle.

Figure 3 depicts how the diagram will look when drawn accurately for $AB \neq AC$. Now, $AR = AS$ and $RB = SC$ still hold, but $AB = AR - RB$ while $AC = AS + SC$ and thus, $AB \neq AC$.

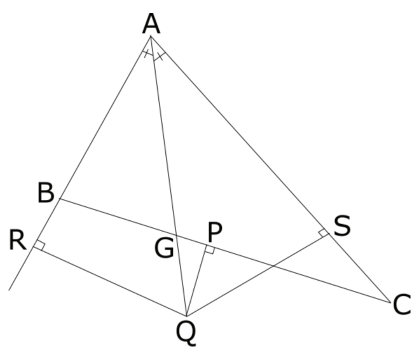

Figure 3.

Both the flaws 'worked' by making an impossible construction look feasible.

To prove that these constructions are impossible, [4] states that Q must lie on the circumcircle of $\triangle ABC$ and so must be outside the triangle. For Flaw #2, a proof is not outlined in [4] (as of this writing).

Our Journey Begins

Treating the resolution as warily as we would a false proof, we asked:

Q1. Let's analyse some available proofs of Q being on the circumcircle.

Q2. Does Q being on the circumcircle guarantee that it lies outside the triangle?

Q3. Can we approach a resolution differently? What about Flaw #2?

Before answering these questions, we need a baseline set of results so that we avoid re-inventing all of Geometry. We choose to use Euclid's *Elements* [2] as our database of 'given' results.

Thoughts on Q1

Several proofs of Q being on the circumcircle assume that Q exists, i.e., the angle bisector and the perpendicular bisector do intersect. In fact, even the false proof assumes this!

Here's a proof that does not make this assumption [3]:

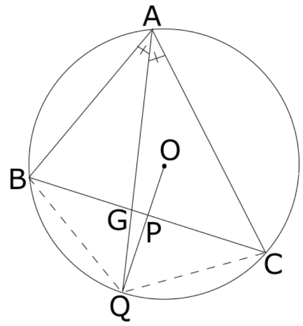

Figure 4.

(1) In Figure 4, extend AG, the bisector of $\measuredangle A$, to cut the circumcircle at Q.

(2) Join QO, where O is the centre of the circumcircle. Let QO cut BC at P.

(3) It is then shown that $\triangle BQP \cong \triangle CQP$ and hence, PQ must be the perpendicular bisector of BC.

Again, the construction looks feasible. What, if anything, would we have questioned had the end result been absurd?

From *Elements*, we know that $\triangle ABC$ and its circumcircle can be constructed, $\measuredangle A$ can be bisected and this bisector AG extended to cut the circumcircle in Q. Finally, given Q and O, we can always construct the segment QO.

But what if QO did not intersect BC?

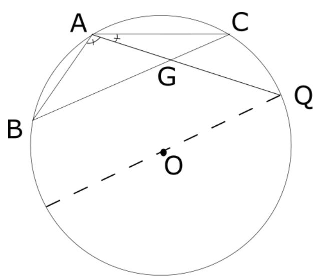

Figure 5.

Exercise 1. Is it possible that for some $\triangle ABC$, $QO \parallel BC$ as indicated in Figure 5?

Thoughts on Q2

Suggesting that a point on the circumcircle could lie inside the triangle seems absurd. But could a circumcircle behave as shown in Figure 6? If yes, then Q could be on the circumcircle AND inside (or on) the triangle! Now, the shape in Figure 6 clearly doesn't look like a circle at all, but how can we prove that it is not?

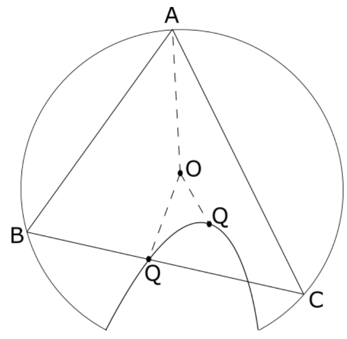

Figure 6.

Actually, for our purposes, proving a somewhat different result is sufficient:

Exercise 2. Show that, for any point Q inside or on $\triangle ABC$, not coincident with the vertices, $OQ < OA$, O being the centre of the circumcircle of $\triangle ABC$. Note: $OA (= OB = OC)$ is the radius of the circumcircle.

Or, we could make it a little more challenging:

Exercise 3. Show that $SQ < \max(SA, SB, SC)$ if S is any point inside the circumcircle.

Exercise 2 is a special case of Exercise 3 and guarantees that any point inside or on the triangle (except the vertices) can't be on the circumcircle.

Thoughts on Q3

Our own examination of the false proof went back to questioning the very existence of Q.

In Figure 7, AG is the angle bisector of $\measuredangle A$ in $\triangle ABC$ and PN is the perpendicular bisector of side BC. If $AB \neq AC$, must AG and PN intersect?

Let's assume $AC > AB$ (the case of $AB > AC$ can be similarly handled). Then, in $\triangle ABC$, $\measuredangle B > \measuredangle C$ (angle opposite larger side is larger).

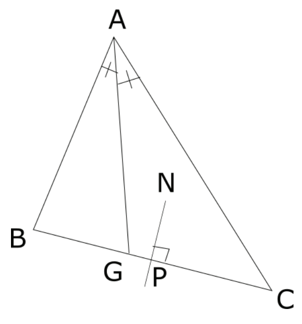

Figure 7.

By simple angle computations, we get: $\measuredangle AGC = B + A/2 > (A + B + C)/2$ since $B > C$. Hence $\measuredangle AGC > 90^\circ$. Thus, $AG \nparallel PN$. Hence AG and PN intersect, meaning, Q exists. But where?

We felt that Euclid's (in)famous Postulate 5 might help answer that question:

Euclid's Postulate 5. *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines (produced), meet on that side.*

In Figure 8, the angle bisector AG is extended to H. We have just seen that $\measuredangle AGC > 90°$; thus $\measuredangle PGH < 90°$. Thus, $\measuredangle PGH + \measuredangle GPM < 180°$. So BC falling on AG and NP is making the interior angles below BC less than two right angles, and by Postulate 5, AG and NP must meet below BC.

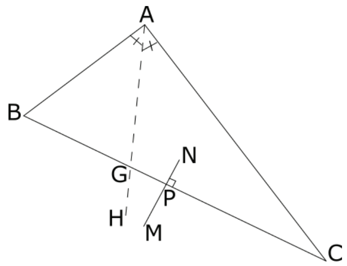

Figure 8.

And now for Flaw #2. We found in [5] a proof addressing Flaw #2. This proof uses 'reflection' and 'symmetry' which we felt, given our 'Element'ary framework, should be simplified a little.That done, we couldn't find anything we would've questioned had this too yielded an absurd result. Can you? The 'simplified' proof is outlined below:

In Figure 9, with $AB < AC$, angle bisector AG of $\measuredangle BAC$ and perpendicular bisector PQ of side BC meet at Q as shown. Since $AB < AC$, we can cut $AE = AB$ on AC and $AD = AC$ on AB-extended.

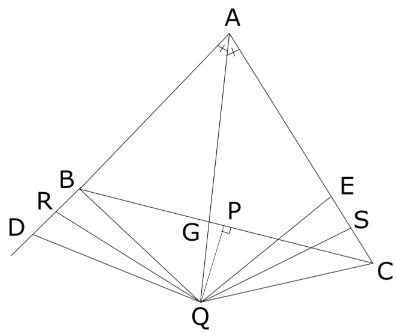

Figure 9.

Now, $\triangle BAQ \cong \triangle EAQ$ by SAS congruence [$AB = AE$ (by construction), $\measuredangle BAQ = \measuredangle EAQ$ (angle bisector) and $QA = QA$ (common)]. Thus, $QB = QE$.

Similarly, $\triangle CAQ \cong \triangle DAQ$ and $QC = QD$.

Further, $\triangle BPQ \cong \triangle CPQ$ by SAS congruence [$BP = CP$, $\measuredangle BPQ = \measuredangle CPQ = 90°$ (perpendicular bisector) and $QP = QP$ (common)]. Thus, $QB = QC$. Thus, we get $QB = QC = QD = QE$.

Thus, $\triangle CQE$ and $\triangle BQD$ are isosceles. From this point, we can continue with the proof exactly as given in [5] and outlined here. The angle bisector of $\measuredangle BQD$ must cut AD in R between B and D, i.e., outside $\triangle ABC$ while the angle bisector of $\measuredangle CQE$ must cut AC in S between E and C, i.e., inside $\triangle ABC$. Finally, we can show that QR and QS are respectively perpendicular to AB and AC.

Conclusion

This exploration taught us that the need for brevity may have hidden interesting results even within correct proofs. Perhaps more importantly, it taught us to watch out for potential oversights when constructing our own proofs.

We invite readers to examine if the proofs we have given have any gaps and how these can be resolved.

References

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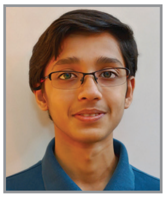

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