Tremendous Tree

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In this article we are primarily interested in constructing what are called "Fraction Trees" and introducing some mathematical concepts behind it. First, we start our exploration from the following construction rule:

Let $a, b > 0$ where a, b are coprime natural numbers. If $\frac{a}{b}$ is a given fraction (proper or mixed) then it generates two terms as shown below

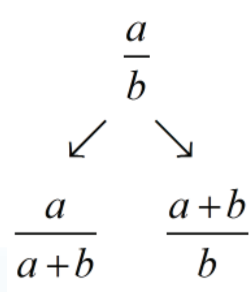

Figure 1.

The term $\frac{a}{a+b}$ is called left child and $\frac{a+b}{b}$ is called the right child of $\frac{a}{b}$. With this convention, we see that $\frac{a}{b}$ is viewed as the parent of both left and right child terms. Also, we see that the left child $\frac{a}{a+b} < 1$. The right child $\frac{a+b}{b} > 1$.

Thus, "Every fraction has two children: a left child, a fraction smaller than 1, and a right child larger than 1."

Keywords: Fractions, iteration

Beginning with the fraction $\frac{1}{1}$ we have the following fraction tree (in this article, we only consider trees beginning with 1):

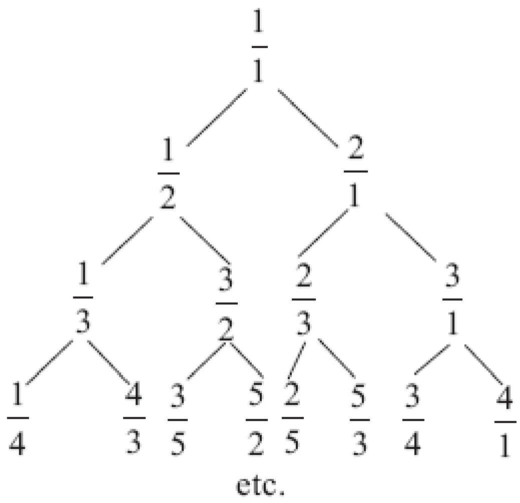

Figure 2.

If we continue drawing another two rows, we get the following tree:

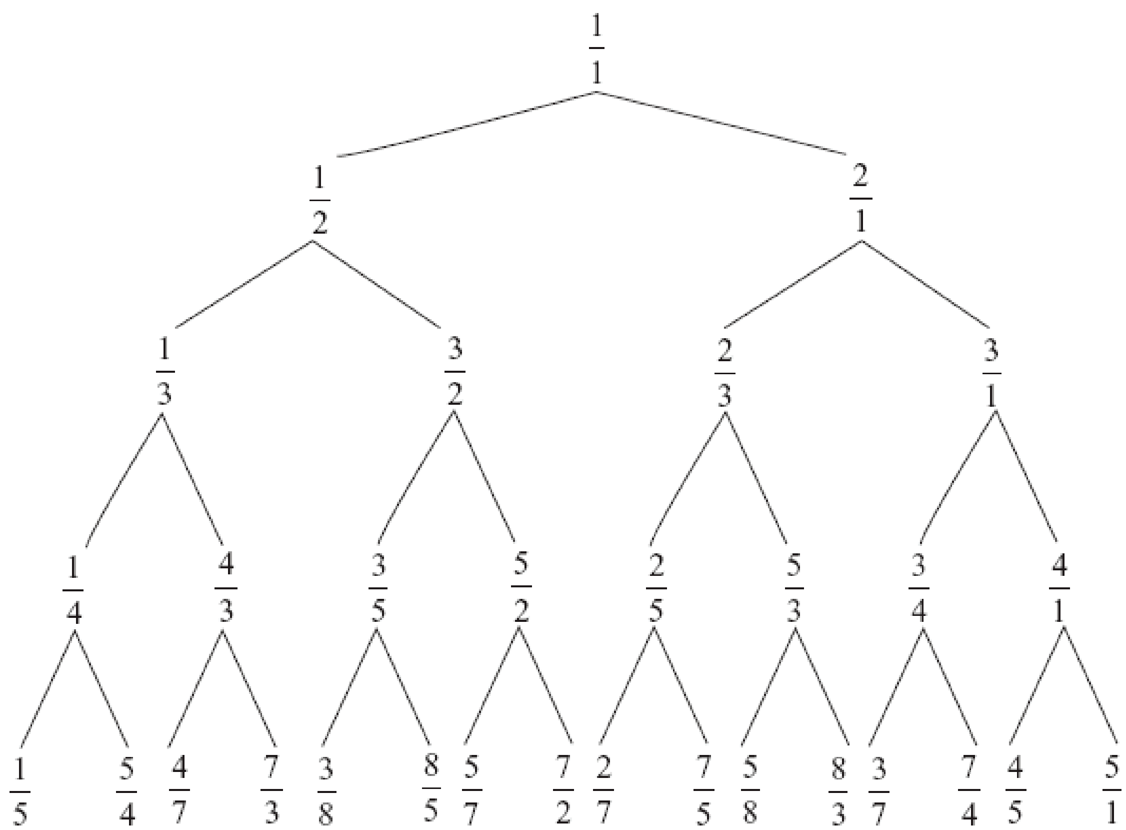

Figure 3.

We see that $\frac{1}{1}$ yields left child $\frac{1}{2}$ and right child $\frac{2}{1}$. Similarly, $\frac{1}{2}$ yields the left child $\frac{1}{3}$ and right child $\frac{3}{2}$. Hence, $\frac{1}{1}$ is considered as grandparent for both $\frac{1}{3}$ and $\frac{3}{2}$. Similarly, $\frac{2}{3}$ is the grandparent for both $\frac{2}{7}$ and $\frac{7}{5}$. In this view, we can consider $\frac{2}{1}$ as the great-grandparent of $\frac{2}{7}$ and $\frac{7}{5}$. (In each case, there may be other grandparents and other grandchildren.) Continuing in this manner and generating more rows in Figure 2, we get several numbers in each generation following the construction rule in Figure 1.

Observing, we will try to answer the following questions:

Challenge I

1. Does the number $\frac{22}{7}$ appear in the tree?
2. If so, what will its parent and grandparent be and how many times does it appear?
3. Did you see any pattern among numerators and denominators in each row of the fraction tree?
4. What is the sum of numerators and denominators of the fractions appearing in each row of the tree?

Many such interesting questions may be asked regarding the fraction tree constructed above. In this article, I will provide a hint for answering some questions regarding this tree and leave the rest to our enthusiastic readers.

Geometer's Algorithm. The ancient Greek mathematician Euclid (hailed as the 'Father of Geometry'), provided a wonderful technique which is now named in his honour as "Euclidean Algorithm" to find the greatest common divisor of two given integers.

He observed that if the greatest common divisor (GCD) of a and b is d where $a > b$, then the greatest common divisor of b and $a - b$ will also be d. By using this principle repeatedly, we arrive at the final pair (d, d) after finitely many steps as both a and b are finite. So, the Algorithm terminates if we get equal numbers in the pair and that number will actually be the GCD of the given numbers.

As an illustration, if we want to determine GCD of 48 and 132, we get

$$(48, 132) \rightarrow (48, 84) \rightarrow (48, 36) \rightarrow (12, 36) \rightarrow (12, 24) \rightarrow (12, 12).$$

Thus 12 is the GCD of 48 and 132. But what, if anything, has this technique to do with our fraction tree?

One of the most curious aspects of mathematics is its ability to connect completely different ideas unexpectedly. Here we are going to witness one such case.

We will try to determine the GCD of the numbers 8 and 5 using the Euclidean Algorithm.

$$(8, 5) \rightarrow (3, 5) \rightarrow (3, 2) \rightarrow (1, 2) \rightarrow (1, 1).$$

If we look at this, we find that the Greatest Common Divisor of 8 and 5 must be 1. But if we look at the pairs between (8, 5) and (1, 1) we notice that they are (3, 5), (3, 2), (1, 2). Rewriting the pair (h, k) as $\frac{h}{k}$, we observe that the pairs formed when the Euclidean Algorithm is applied may be rewritten as $\frac{8}{5} \rightarrow \frac{3}{5} \rightarrow \frac{3}{2} \rightarrow \frac{1}{2} \rightarrow \frac{1}{1}$. But this is precisely the path from $\frac{8}{5}$ to reach $\frac{1}{1}$ if we trace backwards in our fraction tree, as can be seen in Figure 3. Thus, the parent, grandparent, great-grandparent, great-great-grandparents of $\frac{8}{5}$ are $\frac{3}{5}, \frac{3}{2}, \frac{1}{2}, \frac{1}{1}$, respectively.

Thus, the Euclidean Algorithm helps us to trace the ancestral path in the fraction tree constructed through the rule provided in Figure 1. Using Euclidean Algorithm, we could gain much more information about the fraction tree.

Connections to the Fraction Tree. As just stated, starting at $\frac{a}{b}$ and following the path back to $\frac{1}{1}$ is precisely the path taken by Euclid's Algorithm when applied to the pair (a, b). Because the path yields the final pair (1, 1), the fraction $\frac{a}{b}$ is in its simplest form (i.e., its reduced form), and the greatest common divisor of a and b is 1. That is, every fraction of the form $\frac{a}{b}$ in the tree will be in its simplest form.

It can be shown that every reduced fraction $\frac{a}{b}$ must appear somewhere in the tree. By applying Euclidean Algorithm to a and b, we can locate such a fraction in the tree.

Further, we notice that no fraction appears twice in the tree, because if such a fraction occurs twice, then its parents would occur twice, as would its grandparents, great-grandparents, and so on, all the way up to $\frac{1}{1}$ appearing twice, which is not so since we have only one fraction $\frac{1}{1}$ at the apex of the tree. Hence every reduced fraction appears just once in the tree.

These observations probably help you to find answers for questions 1 and 2 of **Challenge I**.

Extra Mile. If we list the fractions of the tree in Figure 2, from left to right across the rows of the tree we obtain the sequence $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \cdots$ We call this sequence as fraction tree sequence. This list contains all positive fractions. The fact that one can list the rational numbers was first discovered by the great German mathematician Georg Cantor in the 19th century. He accomplished this fact in a different way though. (See references [1], [2].)

Further Explorations in the Fraction Tree. If we list the numerators of fractions appearing in each row of the fraction tree in Figure 2, we get 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, . . . If we try to list the denominators then we get 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, . . . which is the same sequence as that of the numerators offset by one. To explore further properties we try to list the numerators reading from left to right, row wise in the fraction tree in Figure 2.

Row 0: 1

Row 1: 1, 2

Row 2: 1, 3, 2, 3

Row 3: 1, 4, 3, 5, 2, 5, 3, 4

. . . .

We observe that **Row *n*** contains 2^n numbers whose sum is 3^n.

If we now list the denominators row wise reading from left to right in the fraction tree in Figure 2, we get

Row 0: 1

Row 1: 2, 1

Row 2: 3, 2, 3, 1

Row 3: 4, 3, 5, 2, 5, 3, 4, 1

. . . .

We notice that each row of denominators is just a mirror image of the corresponding row of numerators. Hence here too, **Row *n*** contains 2^n numbers whose sum is 3^n.

These observations will help you to answer questions 3 and 4 of **Challenge I**.

Considering the list of numerators 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2 . . ., we observe the following properties:

1. Consider every second term, that is the numbers in the even positions. This gives 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2. . ., the original sequence reappears magically.
2. Each term in the odd position (from the third term onwards) is the sum of its two neighbors. That is, $2 = 1 + 1, 3 = 1 + 2, 3 = 2 + 1, 4 = 1 + 3, 5 = 3 + 2 \ldots$
3. Every third term is even and all other terms are odd.
4. The terms between the ones are palindromes given by 2, 323, 4352534, 547385727583745. . . The digits of these palindrome numbers sum to one less than a power of three.

Challenge II

1. Try to prove the first three properties described above.
2. Can you find the 100^{th} term of the fraction tree sequence?
3. Can you find the 1001^{th} term of the fraction tree sequence?
4. In which row does the number $\frac{22}{7}$ appear in the fraction tree?

Binary Representation and Fraction Tree. We know that every number N can be written in binary representation (base 2) containing only 0's and 1's. For example the number 100 in binary (base 2) is 1100100. Similarly the number 17 in binary is 10001. We notice that if N is even, then the binary representation always ends in 0 and if N is odd, it ends with 1.

We try to write N in base two and count the digits that appear in each block of 1s and 0s in its representation. Suppose N is of the form shown below

$$N = \overbrace{1 \cdots 1}^{a_k}\ \overbrace{0 \cdots 0}^{a_{k-1}}\ \overbrace{1 \cdots 1}^{a_{k-2}} \cdots \overbrace{0 \cdots 0}^{a_1}\ \overbrace{1 \cdots 1}^{a_0}$$

where $a_0 = 0$ if N is even. With this representation we observe the following result.

The N th fraction f_N in the fraction tree sequence is the continued fraction given by
$$f_N = [a_0; a_1, a_2, \cdots, a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_k}}}$$

The expression provided above for f_N is called a Continued Fraction. Thus, the Nth fraction f_N in the fraction tree sequence is the finite continued fraction given above.

For example, the 25^{th} term of the fraction tree sequence can be calculated using the above continued fraction expression. First, we find that 25 in base two is 11001. Hence $a_0 = 1, a_1 = 2, a_2 = 2$. Thus, $f_{25} = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = 1 + \cfrac{1}{\frac{5}{2}} = 1 + \frac{2}{5} = \frac{7}{5}$. We can verify from the last row of the fraction tree in Figure 3, that the 25^{th} term of the fraction tree sequence is indeed $\frac{7}{5}$.

This concept will help you to resolve questions 2 and 3 presented in Challenge II.

By making further investigation, you can discover many properties of this amazing fraction tree. In fact, the fraction tree that we are discussing in this article is called the Stern-Brocot tree in mathematics literature. By framing simple rules of creation as in Figure 1, we could generate as many mathematical properties as possible. This is the real charm in doing mathematics.

Suggestions for Further Exploration. For curious readers, I present the following suggestions through which you can learn much more about the Stern-Brocot tree and its associated properties.

1. By creating a construction rule similar to that given in Figure 1, try to construct a tree beginning with $\frac{1}{1}$. This tree is called the Calkin-Wilf tree.

$$\frac{a}{b}$$

$$\swarrow \quad \searrow$$

$$\frac{a}{a+b} \qquad \frac{a+b}{b}$$

   Try to explore similar properties to those we discussed with respect to the Stern-Brocot tree. See if these two trees share anything common between them.

2. The concept of Stern-Brocot tree is applied in various branches of Science and Technology. In particular, it has profound applications in Graph Theory which concerns the structure of networks and connectivity problems. It is also applied in the study of Binary Trees in Computer Science and Algorithms. Try to know these applications and discover new ideas for the future.

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