Problem Corner | Higher Secondary

Two Famous Series for π

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π is an interesting and mysterious number in mathematics. Mathematicians have been interested in this number for the last 4000 years, and work on it is still going on. Its definition is very simple: it is defined as the ratio of the circumference of a circle to its diameter. It does not matter how big or small the circle is, the ratio stays the same. Its approximate value is 3.14159. In this article, we discuss two famous infinite series for π

Although π is defined from the circle in geometry, it mysteriously appears in many formulas in physics and engineering. Surprisingly, it cannot be expressed as a ratio of two whole numbers; i.e., it is an irrational number. In addition to being irrational, it is also transcendental, which means that it is not a solution of any non-constant polynomial equation with rational coefficients.

Mathematicians have been interested for centuries in finding good estimates for π. Archimedes of ancient Greece computed an extremely accurate value around 250 BC. Beginning with regular hexagons inscribed in a circle and circumscribed about the circle, and doubling the number of sides repeatedly till he obtained a regular polygon with 96 sides, he succeeded in showing that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}, \quad \text{i.e.,} \quad \frac{223}{71} < \pi < \frac{22}{7}.$$

The upper bound in this double inequality (i.e., $\frac{22}{7}$) is widely used in schools as the value of π.

Around 150 AD, the Greek scientist Ptolemy gave the value of π as 3.1416 in his book *Almagest*. The Indian astronomer Aryabhata gave the same value in his book *Aryabhatiya* (499 AD).

Today we know that the value of π is $3.141592653589793238\ldots$ For practical calculations, we need a maximum of 5 places after the decimal point. But in their enthusiasm, mathematicians have calculated its value correct up to many trillions of places using modern computers. Even before the invention of the modern computer, mathematicians were able to calculate its value correct up to many places. For example, the English

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mathematician William Shanks (1812-1882) calculated the value of π up to 707 places in 1874 and he took 15 years to find this value ([4]). Later it was found that this value is correct only up to 527 places. The error was found by another English mathematician D.F. Ferguson in 1944, and Ferguson himself calculated the value of π correct up to 710 places in 1947.

How could the value of π be calculated to so many places correctly without the use of computers? It was because of the discovery of some infinite series for π. Here we will discuss two classical series for π, both found by the Indian mathematician Madhava [5] of Sangamagrama (1350-1425).

The first result is the following:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right).$$

Although Madhava's books are lost, the series is found in the book *Tantrasangraha* of Nilakantha Somayaji, written around 1500 AD. The author has attributed it to Madhava.

Madhava also discovered the infinite series for the sine, cosine and inverse tangent functions. Later these were discovered in Europe by James Gregory (1671) and Gottfried Wilhelm Leibniz (1674). The inverse tangent series is:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \qquad (\text{valid for } -1 < x \le 1).$$

This formula was known as the *Gregory-Leibniz series* and is now referred to as the *Madhava series*.

In the above formula, if we put $x = 1$, we get;

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots,$$

which yields the result quoted above.

Proof. From elementary calculus, we know that

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \tan^{-1} - \tan^{-1} 0 = \frac{\pi}{4}.$$

We also know that for $-1 < x < 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots.$$

It can be derived by doing the implied long division or simply by cross multiplying, or by summing the geometric progression on the right side (its common ratio is $-x^2$). Then,

$$\begin{aligned}\frac{\pi}{4} &= \int_0^1 \frac{dx}{1+x^2} \\ &= \int_0^1 \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\Big|_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.\end{aligned}$$

Hence

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

This is the first series developed for π. But it is not useful for calculation purposes, as it converges very slowly. Leonhard Euler (1707-1783) wrote in 1737 that to get just 50 digits from this series, it would require us to "labor fere in aeternum" (work almost forever).

The second result is the following:

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \cdots\right).$$

We may obtain this result by putting $x = \frac{1}{\sqrt{3}}$ in Madhava's series for $\tan^{-1} x$.

We can also find the series as follows:

$$\int_0^{1/\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^{1/\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0 = \frac{\pi}{6}.$$

Hence,

$$\begin{aligned}\frac{\pi}{6} &= \int_0^{1/\sqrt{3}} \frac{dx}{1+x^2}\\ &= \int_0^{1/\sqrt{3}} \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx\\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\Big|_0^{1/\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \cdots\right).\end{aligned}$$

So

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \cdots\right).$$

This series converges quickly, and the sum of just the first 10 terms correctly gives the first five digits of π. The English mathematician Abraham Sharp (1651-1699) used the first 150 terms of the series in 1699 to calculate the first 72 digits of π.

The second series (above) is found in the Malayalam book *Yuktibhasa* written by Jyesthadeva of the Kerala School of Mathematics in about AD 1530. The author has attributed it to earlier mathematicians Madhava and Nilakantha Somayaji. Madhava used this series to calculate the value of π to 11 digits.

References

1. Beckman, Peter, *A History of* π, New York, St. Martin's (1971).
2. Datta B and Singh A.N., *History of Hindu Mathematics: A Source Book*, 2 Vols. Asia Publishing House, Bombay (1962).
3. Paul J. Nahin, *In Praise of Simple Physics, the Science and Mathematics Behind Everyday Questions*, Princeton University Press (2016).
4. Wikipedia, "Pi" from https://en.wikipedia.org/wiki/Pi
5. Wikipedia, "Madhava of Sangamagrama", from https://en.wikipedia.org/wiki/Madhava_of_Sangamagrama

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Convergence issues

When we deal with infinite series, it is in general wise to check the matter of convergence, else we sometimes obtain absurd results. Accordingly we shall do so for the Madhava-Gregory-Leibniz series for π. Fortunately, this is easy to do, as we show below. The analysis also gives an estimate of the error term if we cut short the computation after a certain number of terms.

By summing the GP, we may verify that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2} + \frac{(-1)^n x^{2n}}{1+x^2}.$$

Therefore, integrating from 0 to 1, we get:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} + (-1)^n \int_0^1 \frac{x^{2n}\,dx}{1+x^2}.$$

This relation is exact—it is an identity for all positive integers n. Let us now estimate the value of the integral on the right side. Since $0 \le x^2 \le 1$ for $0 \le x \le 1$, it follows that $1 \le 1 + x^2 \le 2$ for $0 \le x \le 1$, and therefore that

$$\frac{x^{2n}}{2} \le \frac{x^{2n}}{1+x^2} \le x^{2n} \quad \text{for } 0 \le x \le 1.$$

Hence

$$\int_0^1 \frac{x^{2n}\,dx}{2} \le \int_0^1 \frac{x^{2n}\,dx}{1+x^2} \le \int_0^1 x^{2n}\,dx,$$

and so

$$\frac{1}{2(2n+1)} \le \int_0^1 \frac{x^{2n}\,dx}{1+x^2} \le \frac{1}{2n+1}.$$

Hence the error obtained by cutting short the computation at the $\frac{1}{2n-1}$ term lies between $\frac{1}{2(2n+1)}$ and $\frac{1}{2n+1}$. This implies in particular that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \to \frac{\pi}{4}.$$

The other series may be handled in the same manner.

Box 1. A note on convergence (added by the editors)