Two Problems in Number Theory - Part I

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In this two-part article, we study two number theory problems from the UK Math Olympiad, Round 2, years 2006 and 2003 respectively. We consider the first of these problems in Part I. Both problems were discussed during meetings of the problem-solving group of our school.

Problem 1. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y such that

$$x^2 - y^2 = 2^k \tag{1}$$

for some positive integer k.

Solution 1. The problem asks us to find all x and y with no prime factors larger than 5 such that the difference of their squares is some power of 2. There are two things to be dealt with. The first is to find x and y. After that, we must select x and y such that they do not contain prime factors larger than 5. Let us start with the first task.

First, observe that $k \geq 2$. For, as k is a positive integer, 2^k is even, so x, y are both odd or both even. But in this case both $x + y$ and $x - y$ are even numbers, so their product is a multiple of 4.

In the analysis below, we will assume that both x and y are odd. There is no loss of generality in doing so, because if x and y are both even, we can replace them by $x/2$ and $y/2$ respectively (and replace k by $k - 2$), and we can continue doing this until both x and y are odd.

We shall now make use of the fact that a divisor of a power of 2 must itself be a power of 2; no other prime can enter into the factorisation. We have:

$$x^2 - y^2 = 2^k,$$
$$\therefore \quad (x + y)(x - y) = 2^k.$$

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Therefore we must have

$$x + y = 2^a, \quad x - y = 2^b, \tag{2}$$

for some integers a, b, where $a > b \geq 1$. (Note that $a > 1$.)

Solving this pair of simultaneous equations for x and y, we get:

$$x = 2^{a-1} + 2^{b-1},$$
$$y = 2^{a-1} - 2^{b-1}.$$

Since x, y are odd, we must have $b = 1$. Hence:

$$\begin{cases} x = 2^{a-1} + 1, \\ y = 2^{a-1} - 1. \end{cases} \tag{3}$$

From the above, we see that x and y are a pair of consecutive odd numbers.

Having obtained expressions for x and y, we now focus on the requirement that x and y should not be divisible by primes larger than 5.

The only prime factors available are 3 and 5, as x and y are odd. But as $x - y = 2$, one out of x and y must have only 3 as a prime factor, and the other one must have only 5 as a prime factor. That is, one out of x and y is a power of 3, and the other is a power of 5. So the following two cases arise.

Case 1: $x = 3^m$ and $y = 5^n$ for some positive integers m, n. This means that

$$3^m = 2^{a-1} + 1,$$
$$5^n = 2^{a-1} - 1.$$

The second relation is not possible for $a > 2$, as the quantity on the right side is $-1 \pmod 4$ for $a > 2$, whereas the quantity on the left side is $1 \pmod 4$ for any positive integer n.

Hence $a \leq 2$. Since $a > b \geq 1$, we get $a = 2$, and therefore $b = 1$.

Thus, in this case we get $x = 2^1 + 1 = 3$ and $y = 2^1 - 1 = 1$. Note that this corresponds to the solution $3^2 - 1^2 = 2^3$.

Case 2: $x = 5^m$ and $y = 3^n$ for some positive integers m, n. This means that

$$5^m = 2^{a-1} + 1,$$
$$3^n = 2^{a-1} - 1.$$

The second relation is not possible for $a > 3$, as the quantity on the right side is $-1 \pmod 8$ for $a > 3$, whereas the quantity on the left side is $1 \pmod 8$ for $n \geq 2$. (The possibility of $n < 2$ does not arise if $a > 3$, as we would then have $3^n < 2^{a-1}$.) Hence $a = 2$ or $a = 3$.

If $a = 2$, then we get $x = 3$, $y = 1$; but this is the same as the solution obtained above.

If $a = 3$, then we get $x = 5$, $y = 3$. Note that this corresponds to the solution $5^2 - 3^2 = 2^4$.

Recall that we had taken x and y to be odd, by dividing by 2 as often as needed. Replacing these factors, we see that the solutions of the given equation are of the following forms:

$$\begin{cases} x = 3 \cdot 2^k, & y = 2^k, \\ x = 5 \cdot 2^k, & y = 3 \cdot 2^k, \end{cases} \tag{4}$$

where k is any non-negative integer.

References

1. https://www.ukmt.org.uk

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