A Problem in Elementary Number Theory

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In this short note, we look for all cases where the sum of a power of 2 and a power of 3 is a perfect square.

Problem

Find all pairs (m, n) of positive integers such that $2^m + 3^n$ is a perfect square.

Solution

Let $2^m + 3^n = k^2$; note that k is an odd number. We shall make use of the following easily proved number-theoretic facts.

(N1) Any perfect square is of one of the forms $3t$, $3t + 1$ (where t is a non-negative integer).

(N2) Any perfect square is of one of the forms $4t$, $4t + 1$ (where t is a non-negative integer).

(N3) An even power of 2 is of the form $3t + 1$, and an odd power of 2 is of the form $3t + 2$ (where t is a non-negative integer).

(N4) An even power of 3 is of the form $4t + 1$, and an odd power of 3 is of the form $4t + 3$ (where t is a non-negative integer).

In the analysis below, we consider separately the cases when m is odd and when m is even.

Case 1: m is odd. Consider the sum $2^m + 3^n$. Making use of (N3), we see that 2^m is of the form $3t + 2$. As 3^n is a multiple of 3, it follows that $2^m + 3^n$ is of the form $3t + 2$. But no perfect square has this form. Hence the stated equality is not possible. So there is no solution where m is odd.

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Case 2: m is even. We first show that in this case n itself is even.

Suppose that n is odd. Consider the sum $2^m + 3^n$. Making use of (N4), we see that 3^n is of the form $4t+3$. Also, 2^m is a multiple of 4, as m is even. Hence $2^m + 3^n$ is of the form $4t+3$. But no perfect square has this form. Hence the stated equality is not possible. So there is no solution where n is odd.

So we need to only consider the case when both m and n are even.

Let $m = 2a$, $n = 2b$, where a, b are positive integers. We write:

$$2^{2a} + 3^{2b} = k^2,$$
$$\therefore\ 2^{2a} = k^2 - 3^{2b},$$
$$\therefore\ 2^{2a} = \left(k - 3^b\right) \cdot \left(k + 3^b\right).$$

As the quantity on the left side is a power of 2, it follows that both the factors on the right side are powers of 2. Let

$$k - 3^b = 2^c, \quad k + 3^b = 2^d,$$

where c, d are non-negative integers ($c < d$, $c + d = 2a$). By subtraction we get:

$$2 \cdot 3^b = 2^d - 2^c,$$
$$\therefore\ 3^b = 2^{d-1} - 2^{c-1}.$$

If $c > 1$, the quantity on the right side would be even. However, the quantity on the left side (i.e., 3^b) is odd. It follows that $c = 1$. Hence we have:

$$3^b = 2^{d-1} - 1.$$

Since $c + d = 2a$ and $c = 1$, it follows that $d = 2a - 1$. This means that $d - 1 = 2a - 2$ is an even number.

We now have:

$$3^b = 2^{2a-2} - 1,$$
$$\therefore\ 3^b = \left(2^{a-1} - 1\right) \cdot \left(2^{a-1} + 1\right).$$

From the equality in the second line, it follows that both factors (i.e., $2^{a-1} - 1$ and $2^{a-1} + 1$) are powers of 3. *But these numbers are consecutive odd numbers.* The only consecutive odd numbers which are both powers of 3 are 1 and 3 (with $1 = 3^0$ and $3 = 3^1$). Hence $a - 1 = 1$, implying that $2a = 4$ and also $d - 1 = 2$, i.e., $d = 3$, which leads to $b = 1$.

It follows that $m = 4$ and $n = 2$.

We conclude that there is just one pair (m, n) of positive integers such that $2^m + 3^n$ is a perfect square; namely, $(m, n) = (4, 2)$. The associated equality in this case is

$$2^4 + 3^2 = 5^2.$$

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