Ghosts of a Problem Past

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Recently a student of one of us (HH) with the surname Lux[1] brought the following interesting problem to class:

Let c_1 and c_2 be two circles intersecting in A and B. Let a straight line be drawn through A, different from AB, intersecting the two circles in M and N (these being the intersection points different from A). Let K be the midpoint of MN, P the intersection point of the angle bisector of $\angle MAB$ with c_1, and R the intersection point of the angle bisector of $\angle BAN$ with c_2. (We take angles to be 'non-oriented.' That is, they lie between 0° and 180°.) Prove that $\angle PKR = 90°$ (see Figure 1).

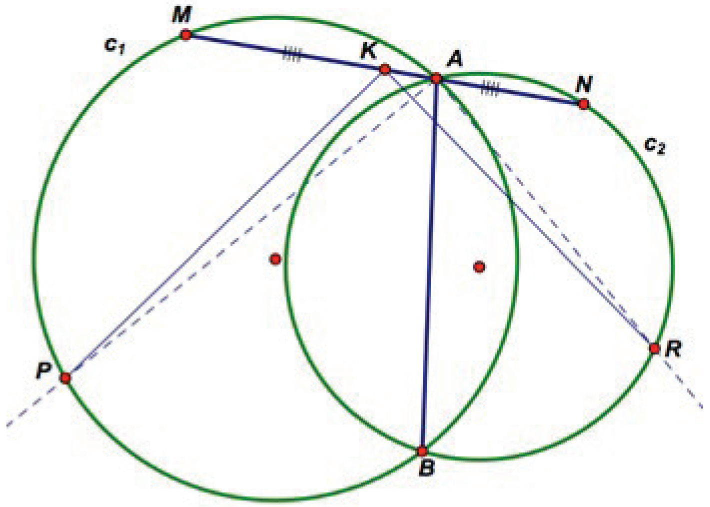

Figure 1: Lux problem

1 The student got the problem from his grandfather, a retired mathematics teacher from France, but without a solution.

Keywords: Intersecting circles, angle bisectors, visualisation, proof

A dynamic, interactive sketch of the Lux problem is available at: http://dynamicmathematicslearning.com/lux-problem.html

The reader is invited to first explore the problem and attempt to prove it before continuing.

Though the problem may be solved using inversion or coordinate geometry, a pure geometry solution proved elusive to find. Despite its elementary appearance, the problem was deceptively hard and resisted several different approaches.

The problem was then shared with MdV who first attempted to prove it using theorems from circle geometry (e.g. trying to prove that quadrilateral *KPRA* is cyclic, etc.), but with no success. It should also be mentioned that though the problem appeared vaguely familiar, MdV was unable initially to make a connection with a past problem, to which we'll come back later. The Lux problem was subsequently shared with several others including a colleague, Waldemar Pompe (WP), from the University of Warsaw, Poland.

After a while, WP came back with a straightforward solution, pointing out that the Lux problem was merely a special case of the following little known but interesting hexagon theorem (see Pompe, 2016, p. 28-29)[2]:

Given a hexagon ABCDEF with AB = BC, CD = DE and EF = FA, and angles α, β, γ *such that* $\alpha + \beta + \gamma = 360°$*, then the respective angles of* $\triangle BDF$ *are* $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ *(see Figure 2).*

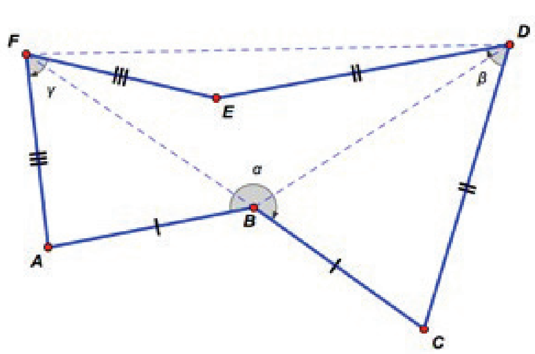

Figure 2: Pompe's Hexagon Theorem

Proof 1 of the Lux problem, using the hexagon theorem

In Figure 1, *AP* and *AR* are the respective angle bisectors of $\angle MAB$ and $\angle NAB$, so the points *P* and *R* respectively bisect the arcs *MPB* and *NRB*; hence *MP* = *PB* and *BR* = *NR* (see Figure 3). Furthermore, it follows from the given that $\angle MPB + \angle BRN + \angle MKN = 360°$. Therefore, the conditions of Pompe's hexagon theorem are met for hexagon *PBRNKM* (at *K* there is a 180° angle!), and it follows that $\angle PKR = 90°$.

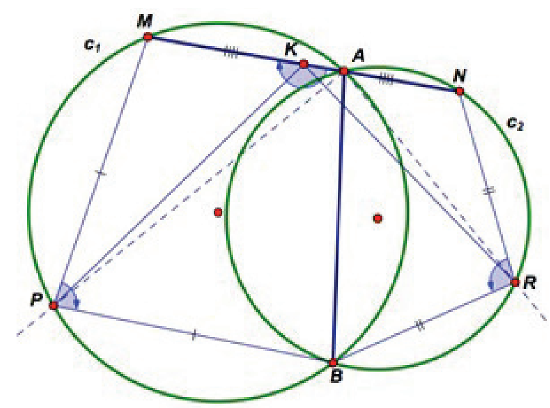

Figure 3: Pompe's Hexagon Proof

Further reflection on Pompe's hexagon theorem reminded MdV of an earlier paper (De Villiers, 2017) involving the sum of rotations, and led to the following proof of the Lux problem.

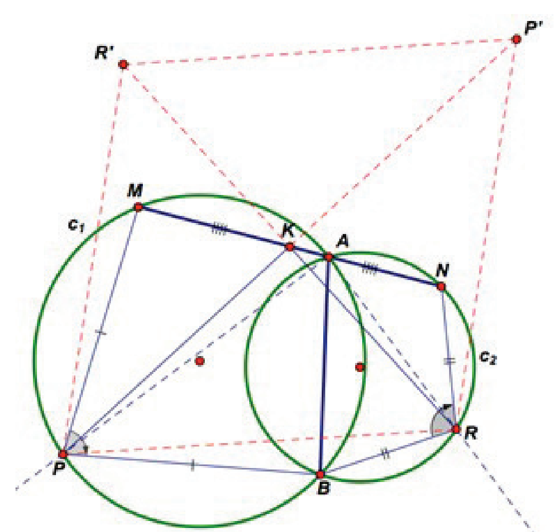

Figure 4: Sum of Two Rotations Proof

2 A proof of Pompe's Hexagon theorem is given in the Appendix. The theorem can also be interactively explored by the reader at: http://dynamicmathematicslearning.com/pompe-hexagon-theorem.html

Proof 2 of the Lux problem, using rotations

Consider Figure 4. Note that a clockwise rotation through ∠*MPB* of *M* around *P*, maps *M* onto *B*, and that a clockwise rotation of *B* through ∠*BRN* = 180° – ∠*MPB* around *R*, maps *B* onto *N*. Therefore, the sum of these two rotations is equivalent to a rotation of 180° around the midpoint *K* of *MN*.

But a counter-clockwise rotation of △*PBR* through ∠*BPM* around *P* and a clockwise rotation of △*PBR* through ∠*BRN* = 180° – ∠*BPM* around *R*, produces a quadrilateral *R'PRP'*. But since angles *BPM* and *BRN* are supplementary, and *R'P* = *RP* = *RP'* from the construction, it follows that *R'PRP'* is a rhombus.

Since △*PMR'* is congruent to △*P'NR* from the earlier rotation of △*PBR*, we now rotate △*PMR'* through a half-turn (180°) around the midpoint of *MN*, namely, *K*, to map onto △*P'NR* with *M* → *N*, *P* → *P'* and *R'* → *R*. But since *R'PRP'* is a rhombus, the only half-turn which will map *P* → *P'* and *R'* → *R* is the one around the "centre" of the rhombus (i.e. intersection point of its diagonals). Therefore *K* must be this centre of the rhombus, and it follows that ∠*PKR* = 90°. (Comment: The proof by Sjoerd Zondervan given in Lecluse (2012) also utilizes the construction of a rhombus, and is very similar to the two rotations proof given here, though not identical.)

Having left the Lux problem for a while before coming back to it later, MdV was reminded of a problem posed by Dick Klingens from the Netherlands at the NVvW annual meeting in November 2011. The problem and several solutions to it were published in the March 2012 issue of *Euclides* (Lecluse, 2012). To our (MdV & HH) surprise, this Klingens problem was identical to the Lux problem!

Ironically, when MdV came across the article by Lecluse during 2012, MdV managed to rather quickly produce an alternative proof involving the nine-point circle, and showing that the problem was really just a special case of a generalization of Van Aubel's theorem involving similar rectangles on the sides (De Villiers, 2013). An interactive, dynamic sketch was also created by MdV in 2013, and was posed as a challenge to mathematically talented students at (with links to relevant papers): http://dynamicmathematicslearning.com/vanaubel-application.html

However, despite this, MdV had completely forgotten about this and did not make the connection until much later. After all, in the process of re-investigating the Klingens-Lux problem, another alternative proof was produced, and it was therefore nonetheless quite productive to revisit these 'ghosts of a problem past.'

Proof 3 of the Lux problem, using angular motion

What follows is a *dynamic* proof because it relies on the *motion of points* (a circle as their orbit and their angular velocity). We found related ideas also in Goddijn (2012) and now we will present a sort of mixture of our and Goddijn's ideas, so that the proof is as short and clear as possible. Our own way had more steps and was more complicated to communicate, but with the help of Goddijn things become more straightforward. This proof will be longer than proofs 1 and 2, but the beauty of this proof lies in its use of *dynamic* issues.

To prepare for the proof we need two lemmas.

Lemma 1: As the point *M* moves on c_1 the point *N* moves on c_2 with the same angular velocity.

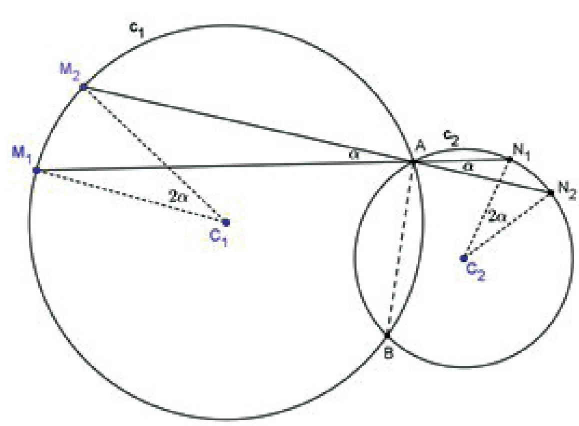

Figure 5: Equal angular velocities of *M* and *N*

The proof follows immediately from Figure 5 and the inscribed angle theorem.

Lemma 2: If two points M, N are moving on circles (centres C_1 and C_2) with equal angular velocities, then also their midpoint K moves on a circle with the same angular velocity, and the centre of this circle is the midpoint D of C_1 and C_2 (see Figure 6).

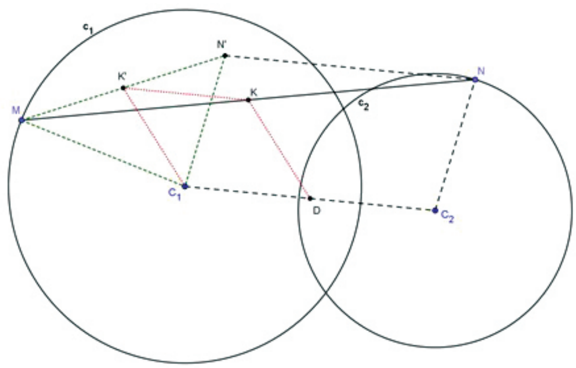

Figure 6: The orbit of K is a circle centred at D

In Figure 6 the point N' is chosen such that C_1C_2NN' is a parallelogram and K' is the midpoint of MN'. Since M and N have the same angular velocity, the angle MC_1N' in the triangle MC_1N' is fixed; in other words the motion of the triangle MC_1N' is a rotation around C_1 with the angular velocity of M on c_1. Because DC_1 and KK' are parallel and equal, we can conclude that C_1DKK' is a parallelogram, too. And from the fact that K' makes a rotation around C_1 it follows that K performs a rotation around D with the same angular velocity, and this completes the proof of Lemma 2.

Now, we know that P and R are moving on their circles with *equal* angular velocity, because their angular velocity is *half* the angular velocity of M and N (P and R come from the angle bisectors!) and M and N do have equal angular velocity (see above). And with Lemma 2 it follows that also the midpoint L of P and R executes a rotation around D with the same angular velocity as P and R (half the one of K; see Figure 7). It is clear that the circle of K passes through A and B, because A is a possible position of K, and we will shortly show that the circle of L passes through C_1 and C_2. Altogether this means in the end: When M, N, K make a full turn around their respective centres, the points P, R, L make a half turn. And for a full turn of P, R, L the points M, N, K need two full turns.

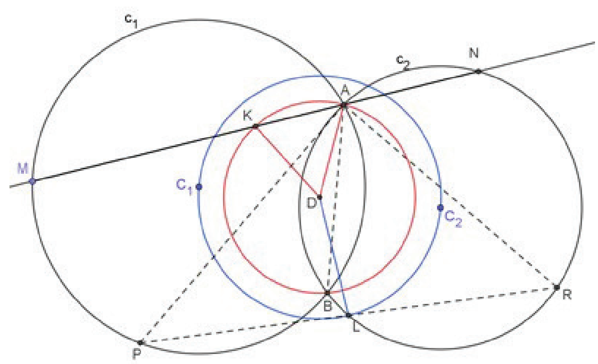

Figure 7: K and L lie on concentric circles centred at D

What happens if M is rotated counter clockwise towards B? Then M, P, N, and K coincide in B (Figure 8[3]).

In the situation $M = B$[4] it is clear that L coincides with C_2 because $\angle PAR = 90°$. In Figure 8b we see that $\angle KDA = 2 \cdot \angle LDA$ (LD is perpendicular to the chord AB and an angle bisector of $\angle BDA = \angle KDA$) and the line segment AD stays fixed during all the motions of the points M, N, K, L, P, R. Since the point K always has the double angular velocity of L, it is clear that during the motions the relation $\angle KDA = 2 \cdot \angle LDA$ always holds. For instance, look again at Figure 7 (so

3 An interactive webpage illustrating the corresponding motions of this dynamic proof is available at: http://dynamicmathematicslearning.com/Klingens-Lux-dynamic-proof.html

As it is really instructive to see the dynamic movement, applets in *GeoGebra* & *SketchPad* can also be downloaded at the above page, or directly from: http://dynamicmathematicslearning.com/klingens-lux-dynamic-proof.zip

4 In this particular situation–which is actually excluded, see page 1–the points K and P coincide, therefore the angle PKR is not defined (and, of course, not right). Another exceptional position of M is $M = A$, because in that situation the line from M to A is not uniquely defined. In our *dynamic* approach it is quite natural to take the tangent of c_1 at A in this situation, so to speak we take the limit of MA (secant) as $M \to A$ (tangent).

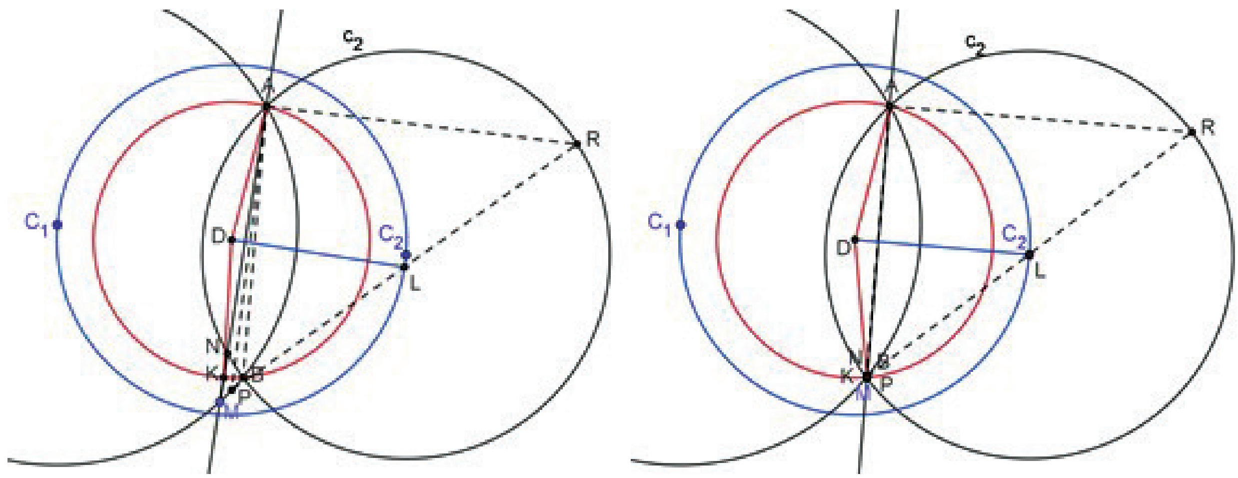

Figure 8: Situation of $M \to B$, shortly before and M = B

to speak we reverse the motions that led from Figure 7 to Figure 8): $\angle LDA \approx 155^\circ$ and $\angle KDA$ is exactly the double. And therefore, *DL* lies on the perpendicular bisector of *AK*, and we can conclude that $KL = AL$ and with Thales' theorem $\angle PKR = 90^\circ$ follows.

Proof 4 of the Lux problem, using similar triangles

Here we present a proof for the Klingens-Lux problem based on the idea by Just Bent (2012). It cleverly makes use of similar triangles in a short and elegant way. To prepare that proof we first formulate the following.

Lemma 3: Let *ABC* be a right triangle, and *CDE* and *BFG* congruent right triangles similar to *ABC* such that $\triangle CDE$ and $\triangle BFG$ are *translations* from each other, i.e., their sides are pairwise parallel. Then the triangle *AFE* is also a right triangle similar to *ABC* (see Figure 9).

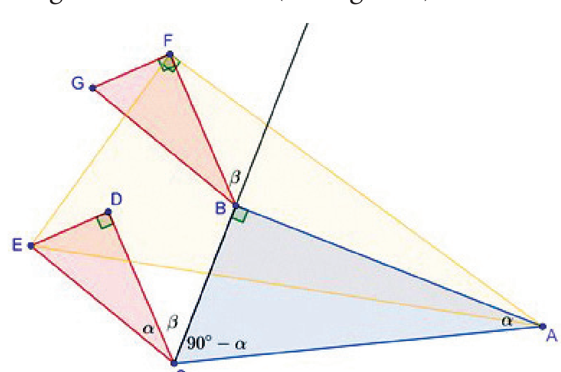

Figure 9: Four similar right triangles

For a proof of Lemma 3, let $\angle CAD = \alpha$ and observe that $\triangle AFB$ and $\triangle AEC$ are similar (they have equal angles at *B* and *C*, namely, $90^\circ + \beta$, and the ratio of the adjacent side lengths is equal: $AB : AC = k = BF : CE$). Therefore, also $AF : AE = k$ and $\angle EAF = \alpha$ hold, and thus the claimed similarity is proven.

And in the Klingens-Lux problem one just has to use this lemma a single time (see Figure 10). In the retrospect things often seem to be very simple, but to *find* these simple relations is sometimes not simple at all; the Klingens-Lux problem is definitely a really hard problem from the perspective of a solver!

Now, for the proof consider Figure 10. Let *H* be the midpoint of *BM, J* the midpoint of *BN*, and $\alpha = \angle BRJ$. Then *KJ* is parallel to *BH* and equal (intercept theorem), all the other angles marked with α, $90^\circ - \alpha$, $180^\circ - 2\alpha$, 2α can easily be derived (cyclic quadrilaterals). And using the above Lemma 3 it follows immediately that $\angle PKR = 90^\circ$.

We have presented here several purely geometric proofs that use completely different means. Proof 1 uses a fairly unknown hexagon theorem, proof 2 uses the fact that the sum of two rotations is a rotation again (the rotation angles add up!)–therefore, this proof could be called a *transformation proof*, proof 3 uses dynamic arguments of motion and could be called a *dynamic proof*, proof 4 uses similar triangles in

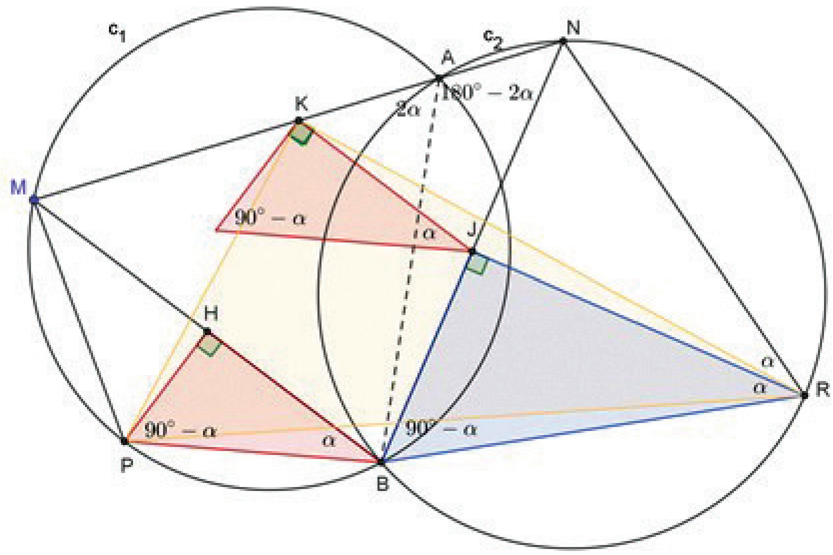

Figure 10: Proof of the Klingens-Lux problem with similar triangles

a smart way, it could be called a *similarity proof*. And still there are many other proofs (see Lecluse 2012 and the references below), so that this problem is a really "rich" one–but one has to admit: quite hard to solve!

Moreover, each of the proofs sheds light in a different way on *why* the result is true; i.e. *explaining* it in a different way. In case of Proof 3 (dynamic) one could also mention the *discovery* function of proof; it was discovered that the points K and L always lie on special circles. This not only illustrates the value of having different proofs for the same result, but also once again, that the value of proof goes far beyond merely that of *verification/conviction*, and that ultimately in mathematics, understanding and insight count for much more.

References

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Glossary of acronyms/abbreviations

- MdV = Michael de Villiers
- HH = Hans Humenberger
- WP = Waldemar Pompe
- NVvW = Nederlandse Vereniging van Wiskundeleraren

Appendix: Proof of Pompe's Hexagon theorem

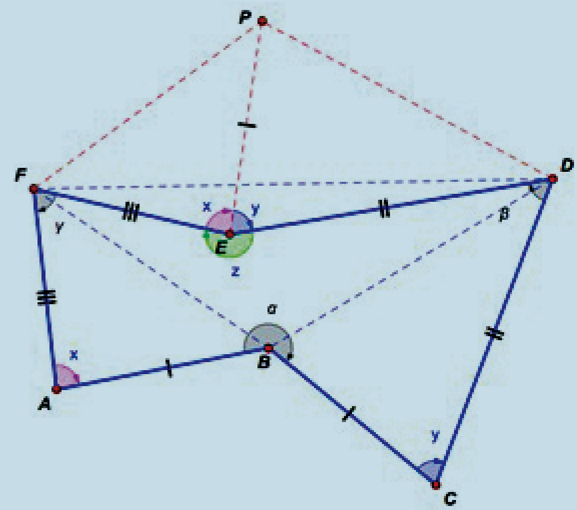


Figure 11: Proof of Pompe's Hexagon Theorem

Proof: Consider Figure 11. Since α + β + γ = 360° is given, it follows that $x + y + z = 360°$. We now have the following possible cases, (a) all x, y and z are less than 180°, or (b) exactly one angle of x, y, z is at least 180°. In (b) we can, without loss of generality, assume $z \geq 180°$. We shall here prove case (b) as the convex case is similar and is left as an exercise to the reader.

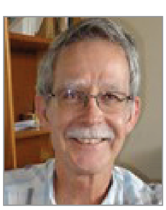
Rotate △*FAB* counter-clockwise around centre *F* through angle γ and △*DCB* clockwise around centre *D* through angle β. Both *A* and *C* map to point *E*, and since $x + y + z = 360°$, it follows that *B′* and *C′* coincide in point *P*.

Since △'s *FEP* and *FAB* are congruent from the rotation, we have ∠*EFP* = ∠*AFB*. Hence, ∠*BFP* = ∠*AFE* = γ. Similarly, it follows that ∠*BDP* = ∠*CDE* = β.

Further, since *FP* = *FB* and *DP* = *DB,* triangles *BDF* and *PDF* are congruent (s, s, s). Therefore, ∠*BFD* = ½ ∠*BFP* = ½ γ and ∠*BDF* = ½ ∠*BDP* = ½ β.

But since ½ α + ½ β + ½ γ = 180°, it follows from the sum of the angles of a triangle in △*BDF* that ∠*FBD* = ½ α. This completes the proof.

Comment: The hexagon theorem of Pompe certainly deserves to be better known as it not only easily proves the Klingens/Lux problem as shown earlier, but also directly applies to 1) proving Napoleon's theorem (the centres of equilateral triangles on the sides of any triangle form another equilateral triangle) as well as 2) immediately showing that in Van Aubel's quadrilateral theorem, the angle formed by the centres of two squares on adjacent sides, say *AB* and *BC*, and the midpoint of the diagonal *AC*, is a right angle.



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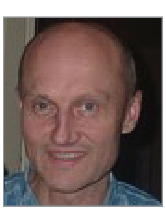

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