An Approach to Cubic Equations

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Suppose we have come to know one root of a cubic equation. What is the quickest way to find the other two roots? In this note, we present a formula for the other two roots. Let the given cubic equation be

$$ax^3 + bx^2 + cx + d = 0, \qquad (1)$$

where $a \neq 0$. Let its roots be u, v, w, and suppose that we have come to know one of them, say w. We derive here a formula for u and v in terms of w and the coefficients a, b, c, d.

Since u, v, w are the roots of the equation, we have

$$ax^3 + bx^2 + cx + d = k(x-u)(x-v)(x-w)$$

for some $k \neq 0$. Expanding the expression on the right and equating coefficients of like powers of x, we get:

$$ax^3+bx^2+cx+d = k\left(x^3 - (u+v+w)x^2 + (uv+vw+wu)x - uvw\right),$$

giving $k = a,\ -k(u+v+w) = b,\ k(uv+vw+wu) = c,$ $-k(uvw) = d$. Hence:

$$u+v+w = -\frac{b}{a}, \quad uv+vw+wu = \frac{c}{a}, \quad uvw = -\frac{d}{a}, \qquad (2)$$

so:

$$u+v = -\frac{b}{a} - w, \quad uv = -\frac{d}{aw}.$$

From these we get:

$$(u-v)^2 = (u+v)^2 - 4uv = \left(\frac{b}{a} + w\right)^2 + \frac{4d}{aw},$$

giving

$$u - v = \pm\sqrt{\left(\frac{b}{a} + w\right)^2 + \frac{4d}{aw}}.$$

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From the expressions for $u+v$ and $u-v$, we get by addition and subtraction,

$$u, v = \frac{1}{2}\left(-\frac{b}{a} - w \pm \sqrt{\left(\frac{b}{a}+w\right)^2 + \frac{4d}{aw}}\right). \tag{3}$$

Thus we obtain the other two roots in terms of w and the coefficients of the equation.

Example. Consider the equation $x^3 + 2x^2 - 4x + 1 = 0$. One of its roots is $w = 1$ (check: $1 + 2 - 4 + 1 = 0$). Here we have $a = 1$, $b = 2$, $c = -4$, $d = 1$, $w = 1$. Therefore:

$$u, v = \frac{1}{2}\left(-2 - 1 \pm \sqrt{3^2 + 4}\right) = \frac{1}{2}\left(-3 \pm \sqrt{13}\right).$$

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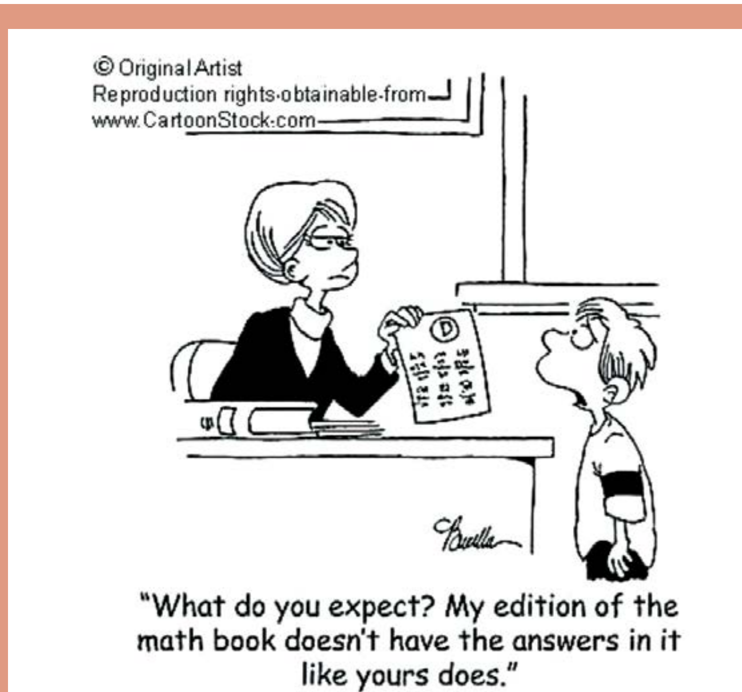

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