Upper Primary

Problem Corner

On Some Questions Related to a Triangle

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A case of two coincident centroids

Consider an acute-angled triangle ABC and let Ω be its circumcircle (Figure 1). Let G be the centroid of ABC. Let the lines AG, BG, and CG meet Ω again at A_1, B_1, and C_1, respectively. Note that if ABC is equilateral, then its centroid G is also the centroid of $A_1B_1C_1$. Suppose it happens that G is the centroid of $\triangle A_1B_1C_1$. Can we then conclude that triangle ABC is equilateral? This is one of the questions we explore in this article.

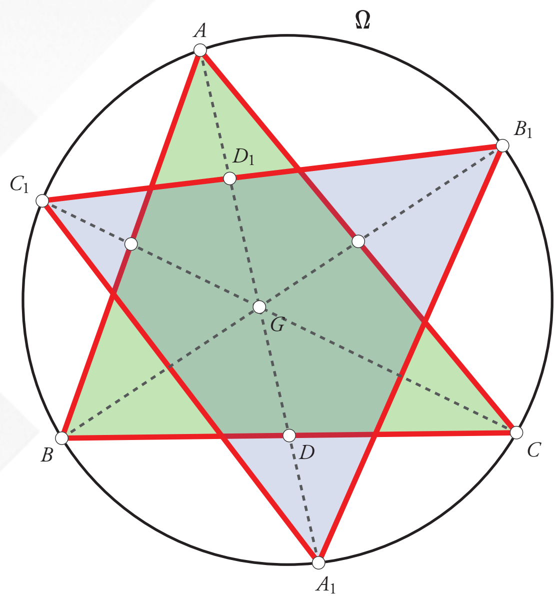

Figure 1. What can be said if G is the centroid of both ABC and $A_1B_1C_1$?

Keywords: Triangle, circumcircle, centroid, incentre, orthocentre, circumcentre, equilateral

Observe that in the triangles BGC and B_1GC_1, $\measuredangle CBG = \measuredangle B_1C_1G$ and $\measuredangle BGC = \measuredangle B_1GC_1$. Therefore they are similar and

$$\frac{BC}{B_1C_1} = \frac{BG}{C_1G}.$$

Let AG meet BC at D, and let A_1G meet B_1C_1 at D_1. Then since D and D_1 are midpoints of BC and B_1C_1, respectively it follows that

$$\frac{BC}{B_1C_1} = \frac{DB}{D_1C_1} = \frac{BG}{C_1G}.$$

Therefore, in triangles DBG and D_1C_1G, we have

$$\measuredangle DBG = \measuredangle D_1C_1G, \quad \frac{DB}{BG} = \frac{D_1C_1}{C_1G},$$

hence they are similar. Hence $\measuredangle DGB = \measuredangle D_1GC_1$. But

$$\measuredangle DGB = \measuredangle B_1GD_1, \qquad \measuredangle D_1GC_1 = \measuredangle DGC.$$

Therefore, we have

$$\measuredangle DGB = \measuredangle D_1GC_1 = \measuredangle DGC = \measuredangle B_1GD_1.$$

This shows that in triangle BGC, GD bisects $\measuredangle BGC$, and in triangle B_1GC_1, GD_1 bisects $\measuredangle B_1GC_1$. Therefore,

$$\frac{BG}{CG} = \frac{BD}{CD} = 1; \quad \frac{B_1G}{C_1G} = \frac{B_1D_1}{C_1D_1} = 1.$$

Hence $BG = CG$ and $B_1G = C_1G$. Similarly, we can prove that $CG = AG$ and $C_1G = A_1G$. Thus, $AG = BG = CG$ and $A_1G = B_1G = C_1G$, implying that the medians of ABC are equal, and so are the medians of $A_1B_1C_1$. Therefore, ABC and $A_1B_1C_1$ are equilateral triangles, and as they have the same circumcircle, they are congruent to each other.

Variations on the theme

One can also explore the cases where instead of the centroids of the two triangles being coincident, the incentres and the orthocentres coincide.

Coincident incentres. Suppose the incentres coincide. Let I be the common incentre of ABC and $A_1B_1C_1$. Then

$$\measuredangle BIC = 90^\circ + \frac{A}{2}, \qquad \measuredangle B_1IC_1 = 90^\circ + \frac{90^\circ - \frac{A}{2}}{2} = 135^\circ - \frac{A}{4},$$

and $\measuredangle BIC = \measuredangle B_1IC_1$ which readily yields $A = 60^\circ$. Similarly, $B = C = 60^\circ$ and ABC is equilateral. Also, $A_1 = 90^\circ - \frac{A}{2} = 60^\circ$ and similarly $B_1 = C_1 = 60^\circ$ showing that $A_1B_1C_1$ is equilateral.

Here we have used the fact that the angles of $A_1B_1C_1$ are $A_1 = 90^\circ - \frac{A}{2}$, $B_1 = 90^\circ - \frac{B}{2}$ and $C_1 = 90^\circ - \frac{C}{2}$. These relations can readily be deduced by angle-chasing.

Coincident orthocentres. Suppose the orthocentres coincide. Let H be the common orthocentre of ABC and $A_1B_1C_1$. Then

$$\measuredangle B_1HC_1 = \measuredangle BHC.$$

But $\measuredangle BHC = 180^\circ - A$ and $\measuredangle B_1HC_1 = 180^\circ - A_1 = 180^\circ - (180^\circ - 2A) = 2A$. Hence

$$2A = 180^\circ - A,$$

and $A = 60^\circ$. Similarly, it follows that $B = C = 60^\circ$ and that ABC is equilateral. So H is also the circumcentre of ABC. Since both ABC and $A_1B_1C_1$ have the same circumcircle, H must also be the circumcentre of $A_1B_1C_1$ as well. But, then the circumcentre and orthocentre of $A_1B_1C_1$ are coincident points implying that $A_1B_1C_1$ is equilateral.

One could have also reached this conclusion by computing the angles of $A_1B_1C_1$ with the help of the expressions

$$A_1 = 180^\circ - 2A, \qquad B_1 = 180^\circ - 2B, \qquad C_1 = 180^\circ - 2C.$$

What if the centroid G of ABC is the incentre of $A_1B_1C_1$? Are both ABC and $A_1B_1C_1$ equilateral? Yes. To prove this, we use the fact that the incentre of $A_1B_1C_1$ is the orthocentre of ABC (a simple exercise for the reader). This shows that the centroid and the orthocentre of ABC are coincident, forcing it to be an equilateral triangle. Since $A = 90^\circ - \frac{A_1}{2}$ we obtain $A_1 = 60^\circ$ and similarly $B_1 = C_1 = 60^\circ$, making $A_1B_1C_1$ equilateral too.

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