Some Results concerning Regular Polygons

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Consider a regular n-sided polygon with centre O and side length a. We first find an expression for its area in terms of a and n. The angle subtended by each side at the centre O of the circumscribing circle is (in radian measure) $2\pi/n$. For convenience, we denote this angle by 2θ.

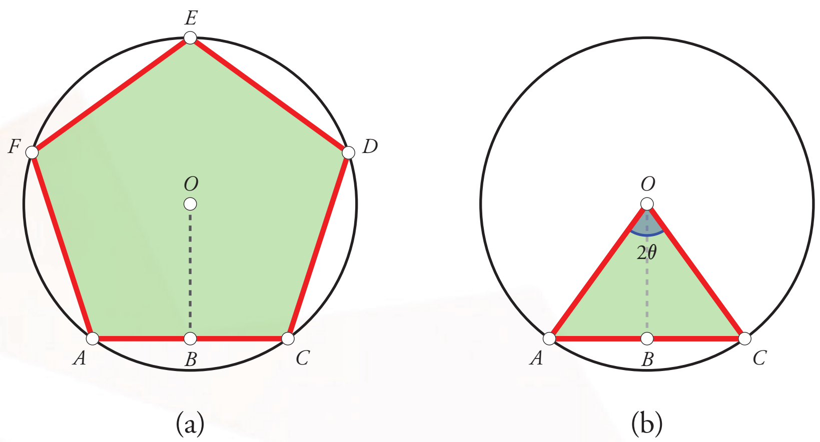

Figure 1. Regular n-sided polygon: $AC = a$, $\measuredangle AOC = 2\pi/n$, $\theta = \pi/n$

If AC is any side of the polygon (see Figure 1), and B is the foot of the perpendicular from O to AC, then $OB = a/2 \cdot \cot\theta$, so:

$$\text{Area of } \triangle OAC = \frac{a^2}{4} \cdot \cot\theta,$$

$$\therefore \text{ Area of polygon} = \frac{n\,a^2}{4} \cdot \cot\theta.$$

Keywords: Regular polygon, incircle, circumcircle, area, perimeter

Incircle and circumcircle of a regular polygon

The incircle. Consider first the incircle of a regular n-sided polygon with side length a. If AC is a side of the polygon (Figure 2a), its point of contact with the incircle being B, then $\measuredangle AOB = \theta$ and $AB = a/2$. A study of $\triangle OAB$ shows that the radius of the incircle is $a/2 \cdot \cot\theta$, so

$$\text{Area of incircle} = \frac{\pi a^2}{4} \cdot \cot^2\theta,$$
$$\text{Circumference of incircle} = \pi a \cdot \cot\theta.$$

Special case. If the numerical values of the circumference and area of the incircle are equal, then

$$\frac{\pi a^2}{4} \cdot \cot^2\theta = \pi a \cdot \cot\theta,$$
$$\therefore \quad a = 4\tan\theta.$$

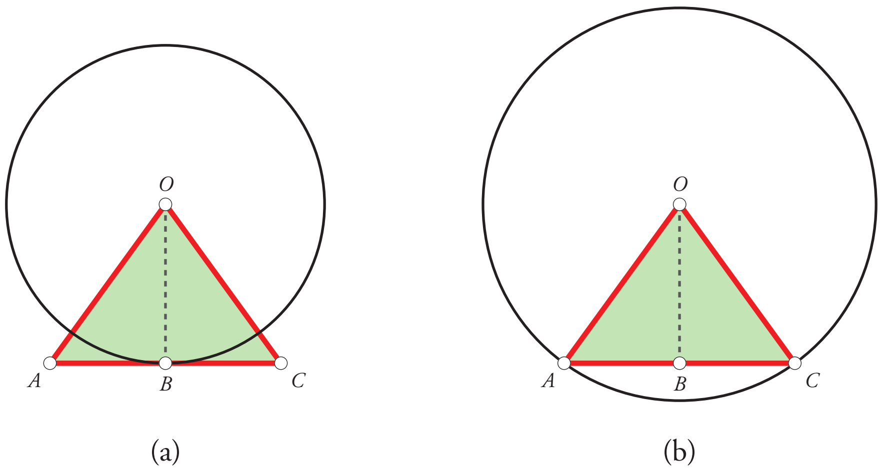

Figure 2.

The circumcircle. Now consider the circumcircle of a regular n-sided polygon with side length a. If AC is a side of the polygon (see Figure 2b), and B is its midpoint, then $BC = OC \cdot \sin\theta$, so

$$OC = \frac{a}{2} \cdot \csc\theta,$$
$$\therefore \quad \text{Circumference of circumcircle} = \pi a \cdot \csc\theta,$$
$$\text{Area of circumcircle} = \frac{\pi a^2}{4} \cdot \csc^2\theta.$$

Special case. If the numerical values of the circumference and area of the circle are equal, then

$$\pi a \cdot \csc\theta = \frac{\pi a^2}{4} \cdot \csc^2\theta,$$
$$\therefore \quad a = 4\sin\theta.$$

Observe that in this situation, the length of the side of such a polygon cannot exceed 4, as the sine of an angle cannot exceed 1.

A remarkable finding. From the above relations, we see that the difference in the areas of the circumcircle and the incircle of a regular n-sided polygon with side length a is equal to

$$\frac{\pi a^2}{4} \cdot \csc^2 \theta - \frac{\pi a^2}{4} \cdot \cot^2 \theta = \frac{\pi a^2}{4}.$$

So the difference in the areas of the circumcircle and the incircle of a regular polygon depends only on the length of its side and not on the number of sides of the polygon. A striking result!

Inscribed and circumscribed regular polygon in a circle

We now consider the reverse situation: we are given a circle with centre O and radius r, and we inscribe in it, and circumscribe about it, regular n-sided polygons.

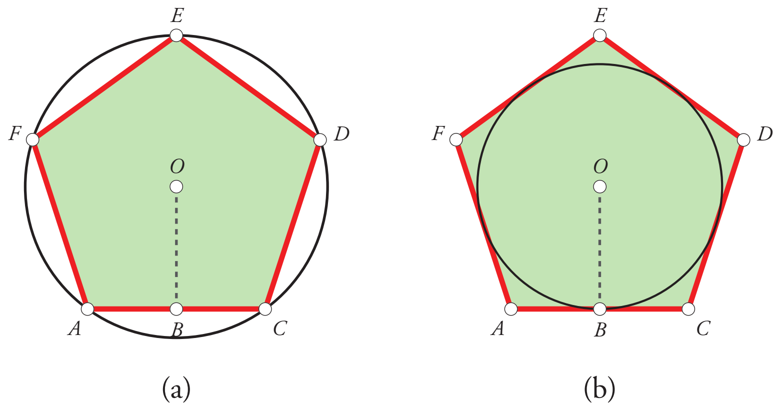

Figure 3.

Inscribed regular polygon. Consider first the inscribed regular polygon. (Recall that 'inscribed' means that the polygon is inside the circle, with all its vertices on the circumference of the circle.) If AC is any side of the polygon, and B is its midpoint (Figure 3a), then $\measuredangle AOB = \theta$, so $AB = r \cdot \sin\theta$. Hence the length of each side of the polygon is $2r \cdot \sin\theta$. Therefore:

$$\text{Perimeter of inscribed polygon} = 2nr \cdot \sin\theta,$$

$$\text{Area of inscribed polygon} = \frac{nr^2}{2} \cdot \sin 2\theta = nr^2 \cdot \sin\theta \cdot \cos\theta.$$

Special case. If the numerical values of the perimeter and the area of the inscribed polygon are equal, then

$$2nr \cdot \sin\theta = nr^2 \cdot \sin\theta \cdot \cos\theta,$$

$$\therefore \quad r = 2\sec\theta.$$

Observe that in this situation, the length of the side of the polygon cannot be less than 2, as the secant of an acute angle cannot be less than 1.

Circumscribed regular polygon. Consider next a circumscribed regular polygon. (Recall that 'circumscribed' means that the circle lies inside the polygon, all of its sides being tangents to the circle.) If AC is a side of the polygon, its point of contact with the circle being B (Figure 3b), then $\measuredangle AOB = \theta$. A study of $\triangle OAB$ yields $AB = r \cdot \tan\theta$. Hence the side of the polygon is $2r \cdot \tan\theta$. It follows that:

$$\text{Perimeter of the circumscribed polygon} = 2nr \cdot \tan\theta,$$

$$\text{Area of the circumscribed polygon} = nr^2 \cdot \tan\theta.$$

Special case. If the numerical values of the perimeter and the area of the circumscribed polygon are equal, then

$$2nr \cdot \tan\theta = nr^2 \cdot \tan\theta,$$

$$\therefore \quad r = 2.$$

Since r is independent of n, it implies that for a circle of radius 2, the area and perimeter of a circumscribed polygon are numerically equal irrespective of the number of sides of the polygon.

Convergence patterns

Consider a regular n-sided polygon with side length a. Let circles be inscribed in it and circumscribed about it. Suppose that we regard the area of the incircle as an approximation for the area of the polygon. In this case, the relative error is

$$\frac{na^2/4 \cdot \cot\theta \; - \; \pi a^2/4 \cdot \cot^2\theta}{na^2/4 \cdot \cot\theta} = \frac{n - \pi \cdot \cot\theta}{n} = 1 - \frac{\pi}{n} \cdot \cot\frac{\pi}{n}.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.21	0.145	0.093	3.3×10^{-2}	3.3×10^{-4}	3.3×10^{-6}

As expected, the relative error decreases with n. (Observe that for large values of n, when n grows by a factor of 10, the relative error shrinks by a factor of 100. This is a very striking pattern.)

Similarly, if we regard the area of the circumcircle as an approximation for the area of the polygon, then the relative error is

$$\frac{\pi a^2/4 \cdot \csc^2\theta \; - \; na^2/4 \cdot \cot\theta}{na^2/4 \cdot \cot\theta} = \frac{\pi \cdot \csc^2\theta - n \cdot \cot\theta}{n \cdot \cot\theta} = \frac{2\pi/n}{\sin 2\pi/n} - 1.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.57	0.32	0.21	0.069	6.6×10^{-4}	6.6×10^{-6}

The relative error decreases with n. (Some interesting patterns may be seen. Once again we note that when n grows by a factor of 10, the relative error shrinks by a factor of 100. Moreover, for each fixed large value of n, the relative error here appears to be twice the relative error in the previous case. This certainly merits further exploration.)

We now invert the situation and regard the area of the polygon as an approximation for the area of the circle. Let us start with the case of the inscribed regular polygon. Here the relative error is

$$\frac{\pi r^2 - nr^2 \cdot \sin\theta \cdot \cos\theta}{\pi r^2} = 1 - \frac{\sin \pi/n \cdot \cos \pi/n}{\pi/n}.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.36	0.24	0.17	0.065	6.6×10^{-4}	6.6×10^{-6}

The relative error decreases with n. Yet again, we see some interesting patterns.

Finally, if we regard the area of the circumscribed polygon as an approximation for the area of the circle, then the relative error is

$$\frac{nr^2 \cdot \tan\theta - \pi r^2}{\pi r^2} = \frac{\tan \pi/n}{\pi/n} - 1.$$

We display below values of the relative error for a few values of n:

n	4	5	6	10	100	1000
Relative error	0.27	0.16	0.10	0.034	3.3×10^{-4}	3.3×10^{-6}

Interesting patterns yet again. The student should take note of these patterns and try to justify them analytically.

Concluding remarks. In this article, a few features of regular polygons inscribed in and circumscribed about circles have been explored. A few striking results have been uncovered, and the idea of using one quantity to approximate another has yielded some interesting patterns which may be explored further by the student.

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