Playing with Quadrilaterals

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In school we are introduced to quadrilaterals as four-sided figures enclosing a region and usually, while discussing their properties, we work only with quadrilaterals each of whose interior angles is less than $180°$. In keeping with this tradition, all quadrilaterals discussed in this article are assumed to have this property.

Let $ABCD$ be a quadrilateral. The diagonals AC and BD intersect at X. Characterise all quadrilaterals $ABCD$ in which

(a) the areas of the triangles ABC, BCD, CDA, and DAB are equal.

(b) the areas of the triangles ABX, BCX, CDX, and DAX are equal.

(c) the perimeters of the triangles ABC, BCD, CDA, and DAB are equal.

(d) the perimeters of the triangles ABX, BCX, CDX, and DAX are equal.

(e) (a) and (c) hold simultaneously.

(f) ((b) and (d)) or ((a) and (d)) hold simultaneously.

(g) (c) and (d) hold simultaneously.

Let us investigate. See Figure 1.

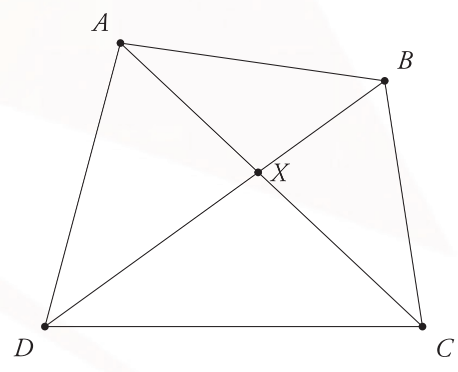

Figure 1.

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(a) The triangles ABC and BCD are on the same base and as their areas are equal, $AD \parallel BC$. Similarly, equality of the areas of triangles ABC and DAB together with the fact that they are on the same base AB imply $AB \parallel DC$. Therefore, $ABCD$ is a parallelogram.

(b) The observation that X is the midpoint of AC as well as BD is immediate which shows that the diagonals of $ABCD$ bisect each other. Hence $ABCD$ is a parallelogram.

(c) Equating the perimeters of the pairs of triangles (BCD, DAB) and (ABC, CDA) leads to

$$DA + AB + BD = BC + CD + BD \tag{1}$$

and

$$AB + BC + CA = AD + DC + CA. \tag{2}$$

Adding (1) and (2) and canceling the common terms yields $AB = CD$. Using this in (1) shows $AD = BC$. Thus the opposite sides of $ABCD$ are equal. Now, by equating the perimeters of the triangles ABD and ABC we get

$$DA + AB + BD = AB + BC + AC \tag{3}$$

which immediately yields $AC = BD$, that is, the diagonals of $ABCD$ are equal. SSS congruence now shows that the triangles CDA and ABC are congruent, and so are the triangles BCD and DAB, leading us to conclude that $ABCD$ is a parallelogram with equal diagonals. Therefore, $ABCD$ is a rectangle (the reader may prove this at leisure!).

(d) Let $AD = x$, $AB = y$, $BC = z$, $CD = w$, $AX = p$, $BX = q$, $CX = r$ and $DX = s$. Equating the perimeters of the four triangles yields

$$x + p + s = y + q + p = z + q + r = w + r + s. \tag{4}$$

But

$$x + p + s = z + q + r = \frac{1}{2}(x + z + p + q + r + s),$$

$$\text{and} \quad y + p + q = w + r + s = \frac{1}{2}(y + w + p + q + r + s),$$

whence

$$x + z + p + q + r + s = y + w + p + q + r + s,$$

which shows

$$x + z = y + w. \tag{5}$$

Thus in quadrilateral $ABCD$ the sums of opposite sides are equal and therefore it has an incircle. In literature, a quadrilateral with an incircle is called a *tangential* quadrilateral or an *inscriptible* quadrilateral. We will return to tangential quadrilaterals later.

(e) $ABCD$ is a rectangle.

(f) $ABCD$ is a rhombus.

(g) $ABCD$ is a square.

Problem 1. *Characterise (if it is possible to characterise!) all quadrilaterals ABCD in which*

(a) the areas of ***some two*** *of the triangles ABC, BCD, CDA, and DAB are equal;*

(b) the areas of ***some two*** *of the triangles ABX, BCX, CDX, and DAX are equal.*

Interestingly, if the areas and perimeters of DAX and CDX are equal, then it follows that $AX = CX$ and $DA + AX = DC + CX$ which gives $DA = DC$, and as DX is the common side of triangles DAX and CDX, by SSS congruence it follows that ADX is congruent to CDX, whence $BD \perp AC$. The triangles ABX and BCX turn out to be congruent by SAS congruence and we get $BA = BC$. Therefore, $ABCD$ is a quadrilateral with BD perpendicular bisector of AC, $DA = DC$ and $BA = BC$. It must be a kite. Note that a kite is also a tangential quadrilateral.

Problem 2. *If AC bisects $\measuredangle DAB$ and $\measuredangle BCD$, what can we say about ABCD?*

Evidently, in this case, the triangles ACD and ACB are congruent by AAS congruence, and we have $AD = AB$ and $CD = CB$. If the diagonals AC and BD intersect at X, then by SAS congruence the triangles AXD and AXB are congruent, and so are the triangles CXD and CXB. Moreover, observe that the diagonals intersect at a right angle. Therefore, $ABCD$ is a kite.

Here is an exercise for the reader.

Problem 3. *If AC bisects $\measuredangle DAB$ and $\measuredangle BCD$, and BD bisects $\measuredangle ABC$ and $\measuredangle ADC$, what type of a quadrilateral is ABCD?*

Tangential quadrilaterals

As promised earlier let us talk about tangential quadrilaterals. A quadrilateral $ABCD$ is tangential if, and only if, $AB + CD = AD + BC$. This is known as Pitot's Theorem and we shall not prove it here.

Suppose $ABCD$ is a tangential quadrilateral and let I be the centre of the incircle. Then, as I is equidistant from the four sides, it lies on the internal bisectors of the four angles of $ABCD$. Therefore, the internal angle bisectors of a tangential quadrilateral are concurrent at the centre of the incircle.

The converse also holds. That is, if the internal bisectors of the angles of a quadrilateral are concurrent, then the quadrilateral is tangential and the point of concurrence is the centre of the incircle.

There is a very simple way to obtain a tangential quadrilateral through a construction. Take a circle and a point outside it. Draw the tangents and join the points of contact to the centre of the circle. The quadrilateral thus obtained is a kite which is tangential. In fact this quadrilateral is cyclic too. See Figure 2.

A quadrilateral which is both cyclic and tangential is called a *bicentric* quadrilateral. Note that in a bicentric kite, the centres of the incircle and the circumcircle lie on one of the diagonals.

In general, is it true that the point of intersection of the diagonals of a bicentric quadrilateral lie on the line joining the centres of its incircle and circumcircle? The reader may indulge in some GeoGebra explorations to find out.

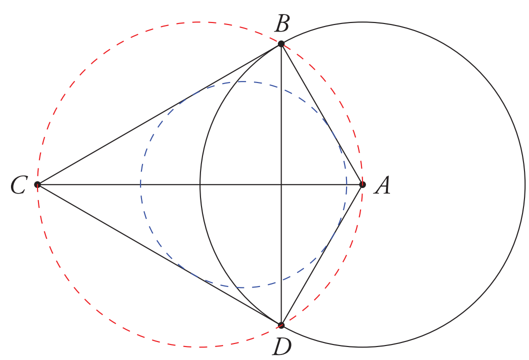

Figure 2. A bicentric kite *ABCD*

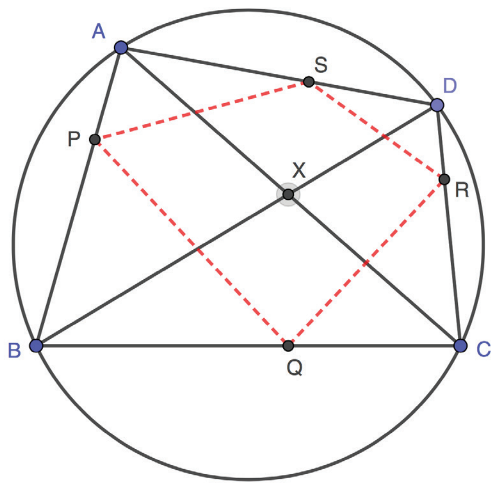

Figure 3. *PQRS* is a tangential quadrilateral

There is a very nice way to obtain a tangential quadrilateral from a cyclic quadrilateral and vice-versa. Let *ABCD* be a cyclic quadrilateral and suppose the diagonals intersect at *X*. If *P*, *Q*, *R*, and *S* are respectively the feet of perpendiculars drawn from *X* onto *AB*, *BC*, *CD*, and *DA* then *PQRS* is a tangential quadrilateral. See Figure 3.

To prove this just observe that the quadrilaterals *XSDR* and *XSAP* are cyclic. This implies $\measuredangle XDR = \measuredangle XSR$ and $\measuredangle XAP = \measuredangle XSP$. But since *ABCD* is cyclic,

$$\measuredangle XDR = \measuredangle BDC = \measuredangle BAC = \measuredangle XAP.$$

Therefore

$$\measuredangle XSR = \measuredangle XSP,$$

that is, *XS* bisects $\measuredangle PSR$. Similarly, it can be shown that *XP* and *XQ* are internal bisectors of $\measuredangle SPQ$ and $\measuredangle PQR$, respectively, showing that *X* is the centre of the incircle of *PQRS*.

Interestingly, if the diagonals *AC* and *BD* intersect at right angles then *PQRS* turns out to be cyclic, and hence bicentric. Maybe the reader can explore to find a proof of this observation.

On the other hand, to obtain a cyclic quadrilateral from a tangential quadrilateral, start with a tangential quadrilateral with mutually perpendicular diagonals and drop perpendiculars from their point of intersection on to the sides . The quadrilateral obtained by joining the feet of the perpendiculars is cyclic. Can you find a proof of this?

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