Bisecting the Perimeter of a Triangle using Ruler and Compass

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In this article, we discuss three geometric construction methods for dividing the perimeter of a given triangle into two equal parts. The methods require only a ruler and a compass.

We may express the problem as follows:

Given an arbitrary triangle, how can we locate pairs of points on the sides that divide the perimeter into two equal parts? (We permit one of the points to be a vertex of the triangle.)

> *Given any polygon and a pair of points on its perimeter, if we travel along the perimeter from one point to the other, we refer to the path as a 'polygonal path.'*

For an equilateral triangle, the midpoint of any one side together with the vertex of the opposite side is one such pair of points, as the length of the path from the midpoint to the opposite vertex (along the perimeter) is the same in both directions.

If the given triangle ABC is isosceles with AB = AC, and D is the midpoint of side BC, then paths DBA and DCA (once again, along the perimeter) have equal length, so the points {A, D} have the required property.

The problem becomes more interesting when the triangle is scalene. We offer three different methods to deal with this case. In the first two methods, one of the points is a vertex of the triangle. In the third method, neither of the points is a vertex of the triangle.

Keywords: Construction, triangle, perimeter, dividing, path length, reasoning

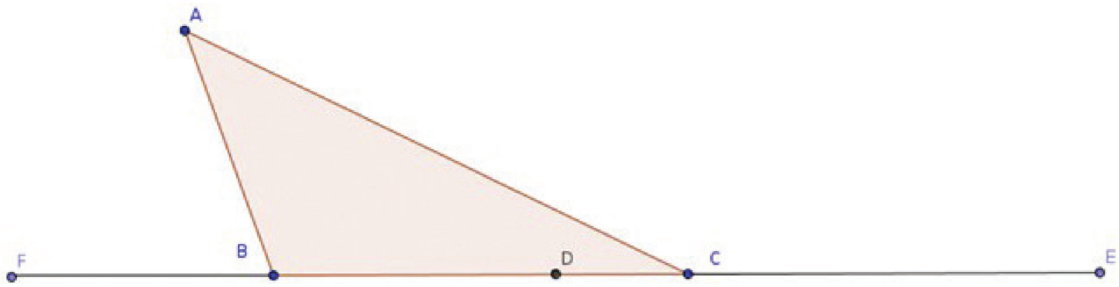

Figure 1

Method 1

We are given a scalene triangle ABC.

Extend BC to point E by length CA and extend CB to point F by length BA (so CE = CA, BF = BA; see Figure 1). Locate the midpoint D of EF, by constructing its perpendicular bisector. Then points A and D divide the perimeter of triangle ABC into two halves. The polygonal paths DCA and DBC have equal length since DC + CA = DE; DB + BA = DF; and DE = DF.

How can we be sure that D lies between B and C? To see why, we use the triangle inequality.

Proof

We have,

$$ED - EC = \frac{(AB + AC + BC)}{2} - AC = \frac{(AB + BC - AC)}{2}.$$

Since AB + BC > AC by the triangle inequality, it follows that ED > EC. This means that D cannot lie between C and E.

Also:

$$EB - ED = \frac{(AC + BC) - (AB + AC + BC)}{2} = \frac{(AC + BC - AB)}{2}.$$

Since AC + BC > AB by the triangle inequality, it follows that ED < EB. This means that D cannot lie between B and F.

Therefore, D lies between B and C.

Method 2

Here we have named the scalene triangle PQR. Construct an escribed circle ('ex-circle') on any side, say side QR. (An ex-circle of a triangle touches one side of the triangle and the extensions of the other two sides, as shown in Figure 2.)

Construction: Extend sides PQ and PR. Construct angle bisectors of the exterior angles at vertices Q and R of ΔPQR. Let the two angle bisectors meet at H. Drop a perpendicular HG from H to QR. Since angles HQR and HRQ are acute, G lies on side QR (and not on its extension on either side).

Since H lies on the angle bisectors of the exterior angles at Q and R of ΔPQR, H is equidistant from PQ extended and QR. Similarly, H is equidistant from QR and PR extended. If we draw a circle with centre H and radius HG, the extensions of PQ and PR will be tangents to the circle. Let the points of tangencies be S and T, respectively.

Since tangents to a circle from a point outside it are equal, PS = PT.

In the same way, {QS, QG} and {RG, RT} are pairs of tangents to the circle from points Q and R, respectively. Therefore QS = QG, and RG = RT. Now we have:

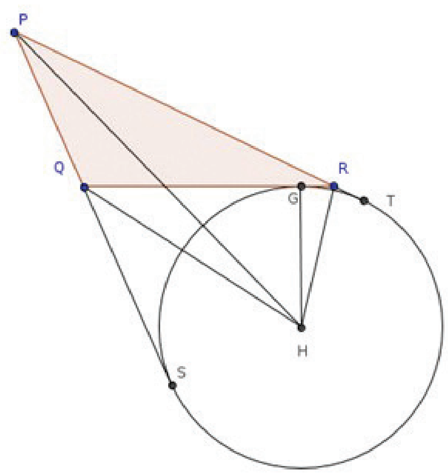

Figure 2

PS = PQ + QS = PQ + QG, and
PT = PR + RT = PR + RG.

Hence PQ + QG = PR + RG.

Therefore, the polygonal paths PQG, PRG (traversed along the perimeter) have equal length.

It follows that points P and G bisect the perimeter of ΔPQR.

Method 3

This time, we have named the scalene triangle XYZ. Assume that XZ > XY (Figure 3).

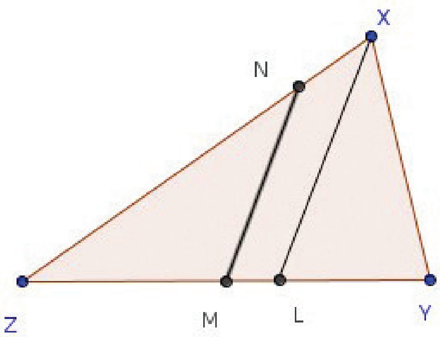

Figure 3

Locate midpoint M of side ZY of ΔXYZ. Let L be the point on ZY where the angle bisector of the opposite angle to side ZY meets side ZY. Draw line through M, parallel to XL. Let this line meet side XZ at N. We shall show below that polygonal path MZN = polygonal path MYXN. Therefore, the points M and N bisect the perimeter of ΔXYZ.

We now justify our claim that ZM + ZN = MY + YX + XN (see Figure 4).

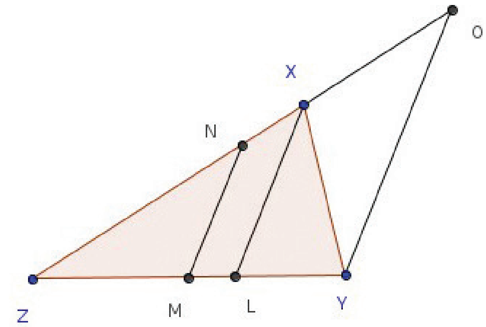

Figure 4

Through Y, draw a line parallel to MN and let it meet ZX extended at O. Since M is the midpoint of ZY and MN is parallel to YO, by the midpoint theorem, N is the midpoint of ZO. Therefore NO = ZN.

MN, XL and YO are parallel lines and ∠XOY = ∠NXL = ∠LXY = ∠XYO. From these relations, it follows that ∠XOY = ∠XYO, so XO = XY. Next: ZN = NO = NX + XO = NX + XY Also, ZM = MY, since M is the midpoint of ZY.

Therefore,

NZ + ZM = NX + XY + YM

Therefore, points M and N bisect the perimeter of triangle XYZ, as claimed.

Remark

The idea behind the third method comes from the broken chord theorem of Archimedes.

References

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