Secondary

Problem Corner

Problems for the Senior School

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Problem IX-3-S.1

Consider the quadratic function $f(x) = x^2 + bx + c$ defined on the set of real numbers. Given that the zeros of f are two distinct prime numbers p and q, and $f(p - q) = 6pq$, determine the primes p and q, and the function f.

Problem IX-3-S.2

Find all positive integers a, b, c satisfying the equation

$$(a+1)^4 \cdot (b+1)^4 \cdot (c+1)^4 = (40a+1) \cdot (40b+1) \cdot (40c+1).$$

Problem IX-3-S.3

In a right-angled triangle ABC, point D lies in the interior of side AC, and point E lies on the extension of hypotenuse AB beyond B. The second intersection of circles ADE and BCE (different from E) is F. Show that $\measuredangle CFD = 90°$.

Problem IX-3-S.4

The areas of two faces of a cuboid are 40 sq.cm and 56 sq.cm. The length of its main diagonal is $\sqrt{138}$ cm. Given that, numerically, the total surface area of the cuboid is a positive integer, determine its volume.

Problem IX-3-S.5

Solve for real x:

$$4^x + 9^x + 36^x + \sqrt{\frac{1}{2} - 2x^2} = 1.$$

Keywords: Quadratic function, cuboid, divisor, multiple

Solutions of Problems in Issue-IX-2 (July 2020)

Problem IX-2-S.1

The numbers 4 and 52 share the following features: both are sums of two squares; both exceed another square by 3. Thus:

$$4 = 0^2 + 2^2, \qquad 4 - 3 = 1^2;$$
$$52 = 4^2 + 6^2, \qquad 52 - 3 = 7^2.$$

Show that there are infinitely many numbers that have these two characteristics. [CRUX]

Solution. We need to show that there are infinitely many integer values of k such that $k = a^2 + b^2$ and $k = c^2 + 3$ for non-negative integers a, b, c. Here is a way of generating such values. Let $a = 2n$, $b = 2n^2 - 2$, and $c = 2n^2 - 1$ for a positive integer n. Then

$$a^2 + b^2 = 4n^2 + 4(n^2 - 1)^2 = 4(n^4 - n^2 + 1),$$
$$c^2 + 3 = (2n^2 - 1)^2 + 3 = 4(n^4 - n^2 + 1).$$

Therefore $a^2 + b^2 = c^2 + 3$ and there is an infinite sequence of numbers

$$k = 4(n^4 - n^2 + 1), \quad n = 1, 2, 3, \ldots$$

with these two characteristics. (The numbers 4 and 52 correspond to $n = 0$ and $n = 2$ respectively.)

Problem IX-2-S.2

Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n. [CRUX]

Solution. Observe that

$$f(n+1) - f(n) = 24 \cdot 25^n - 72 = 24 \cdot (25^n - 3) = 24 \times \text{an even number}.$$

This shows that $f(n+1) - f(n)$ is a multiple of 48 for every positive integer n. Since $f(1) = -48$, which is a multiple of 48, it follows by the principle of induction that $f(n)$ is a multiple of 48 for every positive integer n. This shows that $M \geq 48$. On the other hand, since $f(1) = -48$, M cannot exceed 48. Hence $M = 48$.

Additional comment. Solutions to this problem came from students in Mangalore. We describe them briefly here. It is striking to note the variety of solutions for this problem.

Recall that we need to find the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

Praneetha Kalbavi (Class XI): Let n be any positive integer, and let $f(n) = Mk$ and $f(n+1) = Ml$ for some positive integers k, l; i.e.,

$$25^n - 72n - 1 = Mk, \qquad 25^{n+1} - 72(n+1) - 1 = Ml.$$

We now have:

$$\begin{aligned} Ml = 25^{n+1} - 72(n+1) - 1 &= 25 \cdot 25^n - 72n - 72 - 1 \\ &= 25 \cdot (25^n - 72n - 1) - 72n - 73 + 25 \cdot 72n + 25 \\ &= 25 \cdot Mk - 1728n - 48. \end{aligned}$$

This implies that M is a divisor of $1728n + 48$ for every positive integer n. Since $1728n + 48 = 48(24n + 1)$, it follows that $M = 48$.

Akshar Kumar N (Class XI): First we note $f(1) = -48$ and $f(2) = 480$. Hence $M \leq 48$. Noting that $48 = 16 \times 3$ and $\gcd(16, 3) = 1$, we shall now prove that $3 \mid f(n)$ and $16 \mid f(n)$ for every positive integer n. If we do this, then it will follow that 48 is a divisor of $f(n)$ for every positive integer n, and hence that $M = 48$.

To see why $3 \mid f(n)$, observe that

$$f(n) = 25^n - 72n - 1 \equiv 1 - 0 - 1 \equiv 0 \pmod 3.$$

We prove that $16 \mid f(n)$ by mathematical induction. Note that the hypothesis is true for $n = 1$. Now suppose that $16 \mid f(m)$ for some $m \geq 1$. Let

$$f(m) = 25^m - 72m - 1 = 16k,$$

where k is an integer. We now have:

$$\begin{aligned}
f(m+1) = 25^{m+1} - 72(m+1) - 1 &= 25 \cdot 25^m - 72(m+1) - 1 \\
&= 25 \cdot (16k + 72m + 1) - 72(m+1) - 1 \\
&= 25 \cdot 16k + 72m \cdot 25 + 25 - 72m - 73 \\
&= 25 \cdot 16k + 1800m - 72m - 73 + 25 = 25 \cdot 16k + 1728m - 48 \\
&= 16(25k + 36 - 1),
\end{aligned}$$

proving that $16 \mid f(m+1)$. This justifies the claim that the largest integer dividing $f(n)$ for all n is 48.

Rakshitha (Class XII): Since $f(1) = -48$, it follows that $M \leq 48$. To show that $M = 48$, we first write

$$\begin{aligned}
f(n) &= 25^n - 1 - 72n \\
&= 24 \cdot (25^{n-1} + 25^{n-2} + \cdots + 1) - 24 \cdot 3n \\
&= 24 \cdot (25^{n-1} + 25^{n-2} + \cdots + 1 - 3n) = 24 \cdot g(n),
\end{aligned}$$

where $g(n) = 25^{n-1} + 25^{n-2} + \cdots + 1 - 3n$. Now

$$\begin{aligned}
g(n) &= 25^{n-1} + 25^{n-2} + \cdots + 1 - 3n \\
&\equiv n - 3n \pmod 2 \equiv -2n \pmod 2 \equiv 0 \pmod 2.
\end{aligned}$$

It now follows that $48 \mid f(n)$ for all positive integers n. It follows that $M = 48$.

Problem IX-2-S.3

Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with? [Kvant]

Solution. The answer is eight. Since

$$8 \times 987654321 + 198765432 = 8100000000,$$

the answer is at least 8. But the largest possible value that the sum can take is

$$9 \times 987654321 = 8888888889,$$

so the only other possibility is to have nine zeros.

Now each number whose digits are a permutation of $1, \ldots, 9$ is a multiple of 9, since the sum of their digits is 45 (which is a multiple of 9). Therefore any sum of these numbers must also be a multiple of 9.

But the only 10-digit number ending in nine zeros that is a multiple of 9 is 9000000000, and this is larger than our upper bound.

Problem IX-2-S.4

Note that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$. *Determine conditions for which* $\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}$, *where a, b, c are positive integers.* [CRUX]

Solution. Assume that a, b, and c satisfy

$$\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}, \quad \text{i.e.,} \quad \sqrt{\frac{ac+b}{c}} = a\sqrt{\frac{b}{c}}.$$

Squaring both sides, we get

$$\frac{ac+b}{c} = \frac{a^2b}{c}, \quad \therefore \; ac = b(a^2-1).$$

Since $\gcd(a, a^2-1) = 1$, a cannot divide a^2-1. Therefore a divides b.

Let $b = ka$ for some integer $k \geq 1$. Then $c = k(a^2-1)$.

It is easily verified that for any choice of integers $a \geq 2$ and $k \geq 1$, the triple

$$(a, b, c) = \left(a,\; ka,\; k(a^2-1)\right)$$

will satisfy the condition.

Problem IX-2-S.5

Find all positive integers n satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ *can be split into pairs such that if the numbers in each pair are added, and the sums are then multiplied together, the result is a perfect square.* [Tournament of Towns]

Solution. We claim that n satisfies the condition if $n > 1$. We first observe that $n = 1$ fails the condition. For $n = 1$ the only pairing is $\{1, 2\}$, the sum of which is 3, i.e., not a perfect square. We now consider separately the cases when n is even and when n is odd.

Case 1, n even: Then $n = 2k$ where $k \geq 1$. By pairing i with $2n+1-i$ for $i = 1, 2, \ldots, n$, we get a product of $((2n+1)^k)^2$.

Case 2, n odd: Then $n = 2k+1$ where $k \geq 1$. When $k \geq 1$, we pair 1 and 5, 2 and 4, 3 and 6, and $6+i$ with $2n+1-i$ for $i = 1, 2, \ldots, n-3 = 2k-2$. The product is then

$$(1+5)(2+4)(3+6)(2n+7)^{2k-2} = \left(18 \cdot (2n+7)^{k-1}\right)^2.$$