Definitions changing through time

With a closer look at isosceles trapeziums

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Definitions have changed in textbooks across ages. At one time, squares were not considered rectangles but later they were. Why this change? When we think of classification, we primarily create subsets of a set in a manner that each subset has some specific properties. Essentially there are two ways to do this:

1. **Partition** – where the original set is divided in disjoint i.e. not overlapping subsets.
2. **Hierarchical** – where nested subsets are formed so that a subset with more general properties is the superset of one with a more specific one.

The earlier definition of rectangle, i.e., a parallelogram with a right angle and unequal adjacent sides, followed the Partition classification making it a subset disjoint from squares, which are parallelograms with a right angle and four equal sides (Figure 1). This way of classification makes it easier to depict a shape since it doesn't involve considering various subcases.

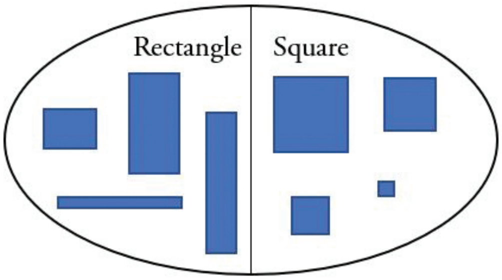

Figure 1: Partition

Keywords: Quadrilaterals, classification, boundaries

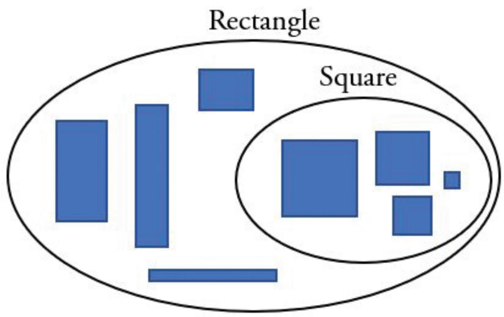

Figure 2: Hierarchy

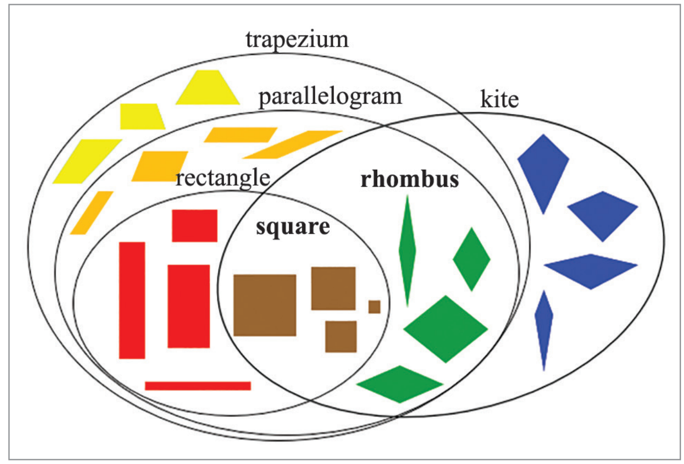

Figure 3

However, the present textbooks have switched to a Hierarchical classification that makes the set of squares a subset of the set of rectangles (Figure 2). A big reason for this change comes from considering the properties of a rectangle. The Partition definition (including unequal adjacent sides) does not provide a rectangle with any property that a square does not have. Or in other words, a square has all the properties of a rectangle. Therefore, it makes sense to consider a square as a (special) rectangle. This does lead to slight complication in depicting such shapes since now subcases may have to be considered e.g. a rectangle can look like a square or like one with unequal adjacent sides.

By the same logic, squares are also rhombi and in fact the intersection of the sets of rectangles and rhombi is precisely the set of squares. Now, the sets of rectangles and rhombi are both subsets of the set of parallelograms, which itself is a subset of the set of trapeziums. Rhombi are also kites. Figure 3 represents these sets. Note that

(i) Rhombus is the intersection of parallelogram and kite i.e. a quad is a rhombus if and only if it is a parallelogram and a kite

(ii) Square is the intersection of rectangle and rhombus i.e. a quad is a square if and only if it is a rectangle and a rhombus

(iii) There is no quad which is a trapezium and a kite but not a parallelogram

We encourage the reader to prove each of the above statements.

But most textbooks are not uniformly adopting this change in definition. By this line of logic, equilateral triangles should be considered special isosceles triangles, but this is not reflected in most textbooks. Isosceles triangles have no property that an equilateral triangle does not have
(See Figure 4). This change however is reflected in some resources available on the web, e.g. https://www.cut-the-knot.org/triangle/Triangles.shtml which includes both cases.

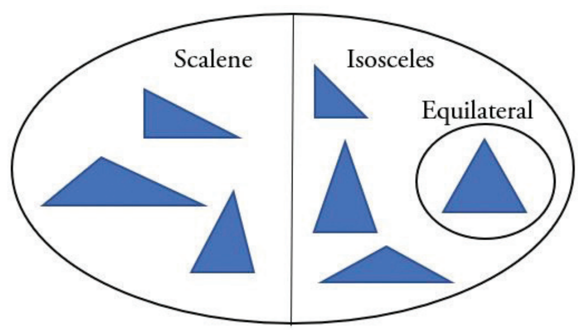

Figure 4

Another interesting case arises w.r.t. rectangles and isosceles trapeziums. Rectangles satisfy all properties of an isosceles trapezium. So, rectangle is the intersection of isosceles trapezium and parallelogram (Figure 5). However, the popular definition of an isosceles trapezium creates a barrier in allowing rectangles to join the set. An isosceles trapezium has a pair of parallel sides and a pair of equal sides. This comes directly from the name 'isosceles' meaning same sides. Now, if the parallel sides are equal, it becomes a parallelogram. So, the standard definition specifies that the non-parallel sides to be equal.

But the non-parallel-ness of the remaining sides excludes rectangles (Figure 6). So, a possible way out is to say, a quad with one pair of parallel sides with the other pair having equal length is an isosceles trapezium. This allows a rectangle to be an isosceles trapezium. But it also leads to one more problem. This alternate definition also allows a parallelogram to be an isosceles trapezium which is not possible. Isosceles trapeziums have line symmetry which a parallelogram need not have. So, there is a problem if we consider only sides:

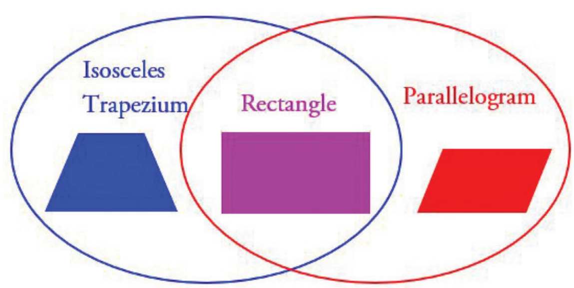

Figure 5

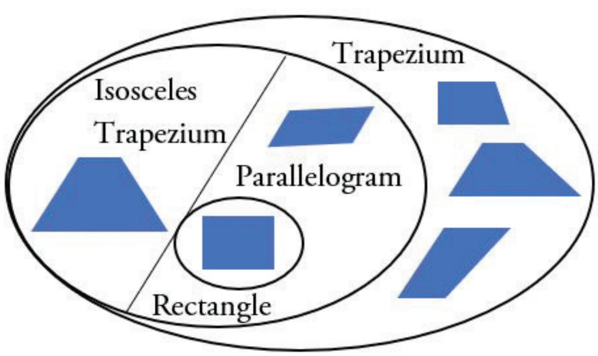

Figure 6

	Desirable	Non-desirable
Other sides are non-parallel	Excludes parallelograms	Excludes rectangles
Other sides can be parallel	Includes rectangles	Includes general parallelograms
Parallel sides are unequal	Excludes parallelograms	Excludes rectangles
Parallel sides can be equal	Includes rectangles	Includes general parallelograms

So, the definitions must go beyond sides and include some statement about the angles. There are several ways to go about this since there are several equivalent conditions:

1. Trapezium with two pairs of equal adjacent angles
2. Trapezium with opposite angles supplementary ⇔ a cyclic trapezium
3. Trapezium with a line symmetry
4. Trapezium with equal diagonals

Each of these includes rectangles but excludes general parallelograms. In the first definition, it is important to mention two pairs since it is possible for a trapezium to have exactly two right angles which must be adjacent. So, just one pair of equal adjacent angles may not ensure an isosceles trapezium. An alternative definition for 1 can be:

5. Trapezium with equal angles adjacent to any of the parallel sides

This excludes the right-angle pair possibility and may be more convenient in terms of construction and pre-requisites compared to 2, 3 and 4. Also it describes the quad in terms of its sides and angles from which properties (like 2, 3 and 4) can be derived, rather than make the derivable property a definition. One may need to understand cyclic quad to grasp the second part of 2 and one must be familiar with the properties of a shape with line symmetry. Whereas, 5 does not require anything beyond sides and angles. If the measure of the equal angles and their common side is given, one can draw the unique isosceles trapezium with these specifications. It involves an ASA type construction and drawing a parallel line. One can thus construct an isosceles trapezium with this definition and then explore its properties. However, it may be more difficult, but not impossible, to construct an isosceles trapezium using 2 or 3. In fact, there is a definition similar to 5 on the web: "An **isosceles trapezoid** (called an **isosceles trapezium** by the British; Bronshtein and Semendyayev 1997, p. 174) is **trapezoid** in which the base angles are equal and therefore the left and right side lengths are also equal." [http://mathworld.wolfram.com/IsoscelesTrapezoid.html]

We end this with the following statement and urge readers to prove it or provide a counterexample: *Any quad with line symmetry is a kite or an isosceles trapezium.*

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PRIME NUMBER GAMES

Prime numbers are an eternal source of fascination for mathematicians, amateurs or otherwise. They offer tantalising glimpses of pattern and symmetry, but these patterns are often elusive and misleading, as many math explorers have found the hard way!

The natural thing to do with prime numbers is to *multiply* them with each other, but mathematicians have also explored what happens when you add prime numbers to one another. There is a famous unsolved problem in this connection, known as Goldbach's conjecture, which states that it is possible to write every even number exceeding 2 as a sum of two prime numbers. For example, 20 = 17 + 3, 30 =17 + 13, 40 = 29 + 11, and so on. This may seem simple to prove – but it has resisted the best efforts of a large number of mathematicians over the past two centuries! (That's why it is called a conjecture.)

We display below some prime number relations found by two budding mathematicians, Harshul and Shresht. Who knows what such number play may lead to? We don't know, and we will not try to guess.

$$(5-3)=(2\times 1),$$
$$(7-5)=(3-2)+1,$$
$$(11-7)\times(3-2)=(5-1),$$
$$(7-5)+(3-2)=(13-11)+1,$$
$$(11-7)\times[(5-3)-(2-1)]=(17-13),$$
$$(7-5)\times[(3-2)+1]=(19-17)+(13-11),$$
$$(11-7)\times(5-3)\times(2-1)-(17-13)+(23-19),$$
$$(29-23)+1=(19-17)+(13-11)+(7-5)+(3-2),$$
$$[(31-29)+(11-7)+(5-3)]\times(2-1)=(23-19)+(17\text{-}13),$$
$$(37-31)+(19-17)+(7-5)=(29-23)+(13-11)+(3-2)+1,$$
$$(41-37)\times[(23-19)-(31-29)]=\{(17-13)[(11-7)-(5-3)]\}\times(2-1);$$

And so on.

And:

$$1\times(2+3)=5,$$
$$(3+7\times 1)\div 5=2,$$
$$(5-3)\times 1+2=11-7,$$
$$(13-7)\times(3-2\times 1)=11-5,$$
$$(17+13)\times 2-(7-3+1)=11\times 5,$$
$$(2\times 3\times 5)+11+7+1=19+17+13,$$
$$23+19+17+13=\{(11\times 7)-5\}\times(3-2\times 1),$$
$$29+23+19+17+13=\{11\times(5+2)\}+\{(7+1)\times 3\},$$
$$31+29+\cdots+13=\{(7+5)\times 11\}\times(3-2\times 1),$$
$$37+31+\cdots+13=(11\times 7\times 2)+(5\times 3\times 1),$$
$$41+37+\cdots+17=11\times(13+2)+(7\times 5)-(3\times 1);$$

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