Problem Corner

Secondary

Partial Sum of Consecutive Integers Equal to a Square

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In this short note, we study the following problem.

> *For which positive integers n is it true that the sum of the positive integers from* 1 *till n is a perfect square?*

Since the sum of the positive integers from 1 till n is $\frac{1}{2}n(n+1)$, the problem may be restated as follows: find all pairs (m, n) of positive integers satisfying the following equation:

$$\frac{n(n+1)}{2} = m^2. \tag{1}$$

To solve this equation, we start with the following simple observations:

- For any positive integer n, the integers n and $n+1$ are co-prime (i.e., they have no factors in common other than 1).
- Precisely one of the integers n and $n+1$ is even.
- If the product ab of two co-prime positive integers a and b is a perfect square, then both a and b are perfect squares.

Hence the following may be stated:

(i) If n is even, then $\frac{1}{2}n$ and $n+1$ are co-prime integers, so if $\frac{1}{2}n(n+1)$ is a perfect square, then both $\frac{1}{2}n$ and $n+1$ are perfect squares.

(ii) If n is odd, then n and $\frac{1}{2}(n+1)$ are co-prime integers, so if $\frac{1}{2}n(n+1)$ is a perfect square, then both n and $\frac{1}{2}(n+1)$ are perfect squares.

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It follows that there are two categories of positive integers n for which the sum of the positive integers from 1 to n is a perfect square, namely:

(i) n is even and both $\frac{1}{2}n$ and $n+1$ are perfect squares. This means that we have $n = 2x^2$ and $n+1 = y^2$ for some positive integers x and y.

(ii) n is odd and both n and $\frac{1}{2}(n+1)$ are perfect squares. This means that we have $n = x^2$ and $n+1 = 2y^2$ for some positive integers x and y.

In case (i) we have $y^2 - 2x^2 = 1$, and in case (ii) we have $x^2 - 2y^2 = -1$. Observe that both these equations are of the following kind:

$$u^2 - 2v^2 = \pm 1, \tag{2}$$

where u and v are positive integers. So we must solve equation (2) over the positive integers.

This is a familiar equation; we have met it many times in the past. One way of generating the solutions is to consider the powers of the irrational number $1+\sqrt{2}$. To be specific, let k be any positive integer, and let the quantity

$$\left(1+\sqrt{2}\right)^k$$

simplify to $u + v\sqrt{2}$, where u and v are integers. Then we may show that $u^2 - 2v^2 = \pm 1$, thus providing a solution to (2). Moreover, *every* solution to (2) may be obtained in this manner, simply by giving different values to k. For example,

- $k = 1$ yields $u = 1$ and $v = 1$. Here $u^2 - 2v^2 = -1$, so we have $n = u^2 = 1$. This corresponds to the not-particularly-interesting relation $1 = 1^2$.
- $k = 2$ yields $(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$, i.e., $u = 3$ and $v = 2$. Here $u^2 - 2v^2 = 1$, so we have $n = 2v^2 = 8$. This corresponds to the first interesting instance of the property we are looking for:
$$1 + 2 + 3 + \cdots + 7 + 8 = \frac{8 \times 9}{2} = 36 = 6^2.$$
- $k = 3$ yields $(1+\sqrt{2})^3 = 7 + 5\sqrt{2}$, i.e., $u = 7$ and $v = 5$. Here $u^2 - 2v^2 = -1$, so we have $n = u^2 = 49$. This corresponds to the relation
$$1 + 2 + 3 + \cdots + 48 + 49 = \frac{49 \times 50}{2} = 1225 = 35^2.$$

Proof that the procedure works. The most effective way of showing that this procedure invariably yields a solution is through induction. However, on this occasion we opt to use a non-inductive approach

As is well-known, if we have

$$(1+\sqrt{2})^k = u + v\sqrt{2}, \tag{3}$$

where u, v are integers, then we also have

$$(1-\sqrt{2})^k = u - v\sqrt{2}. \tag{4}$$

The most direct approach now is to make use of the following two facts:

$$(1+\sqrt{2}) \cdot (1-\sqrt{2}) = -1,$$

and

$$(u + v\sqrt{2}) \cdot (u - v\sqrt{2}) = u^2 - 2v^2.$$

We now get, from (3) and (4):

$$u^2 - 2v^2 = (-1)^k = \pm 1,$$

as required.

A slightly more cumbersome approach is to find explicit expressions for u and v in terms of k, and then to work with those expressions. For convenience, we write $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. From (3) and (4), we obtain, by addition and subtraction respectively:

$$u = \frac{\alpha^k + \beta^k}{2},$$

and

$$v = \frac{\alpha^k - \beta^k}{2\sqrt{2}}.$$

Hence:

$$\begin{aligned} u^2 - 2v^2 &= \frac{\alpha^{2k} + 2(\alpha\beta)^k + \beta^{2k}}{4} - \frac{\alpha^{2k} - 2(\alpha\beta)^k + \beta^{2k}}{4} \\ &= (\alpha\beta)^k = (-1)^k, \quad \text{since } \alpha\beta = -1. \end{aligned}$$

Though this approach is, as noted, more cumbersome, it has the advantage of yielding explicit expressions for u and v in terms of k.

Every solution? It remains to be shown that the above procedure generates every possible solution to the given problem. As one may expect, this part is more challenging.

One approach is to use the above idea in reverse. This enables us to generate solutions using smaller numbers from solutions using large numbers. To make this more clear, we consider the identity

$$(u + v\sqrt{2}) \cdot (1 + \sqrt{2}) = (u + 2v) + (u + v)\sqrt{2}.$$

This shows that from a solution $(x, y) = (u, v)$ to the equation $x^2 - 2y^2 = \pm 1$, where u, v are positive integers, we may generate another positive integral solution $(x, y) = (u + 2v, u + v)$, and this clearly features larger numbers than the original solution. For example, starting with the solution $(x, y) = (1, 1)$ and iterating the map $(u, v) \longmapsto (u + 2v, u + v)$, we obtain the following infinite chain of solutions:

$$(1, 1), \quad (3, 2), \quad (7, 5), \quad (17, 12), \quad (41, 29), \quad (99, 70), \quad \ldots.$$

Now we apply this idea *in reverse*. Write $u' = u + 2v$ and $v' = u + v$. Then clearly:

$$u = 2v' - u', \quad v = u' - v'.$$

From this we infer that given a solution $(x, y) = (u, v)$ to the equation $x^2 - 2y^2 = \pm 1$, where u, v are positive integers, we may generate another solution $(x, y) = (2v - u, u - v)$, and this features *strictly smaller numbers* than the original solution.

Now note the following (here u, v are non-negative integers):

- If $u^2 - 2v^2 = \pm 1$ and $v > 1$, then $2v > u > v > 1$.
- If $u^2 - 2v^2 = \pm 1$ and $v > 1$, then $(u, v) > (1, 1)$ and also $(2v - u, u - v) > (1, 1)$.

Reasoning in this manner, we see that starting with any solution to the given equation $x^2 - 2y^2 = \pm 1$ and iterating the map described above, $(u, v) \longmapsto (2v - u, u - v)$, we obtain a decreasing sequence of solutions. As it is not possible to have an infinitely long strictly decreasing sequence of positive integers, it must happen at some stage that we reach a solution with $v = 1$, which means that we have reached the solution $(1, 1)$, at which point the decrease necessarily comes to a halt.

Reversing the map again, we infer that every solution to the equation belongs to the chain shown above:

$$(1,1), \quad (3,2), \quad (7,5), \quad (17,12), \quad (41,29), \quad (99,70), \quad \ldots.$$

But this implies that the procedure we have described does generate every possible solution to the problem. No solution is missed out.

Recalling the connection between the integer pairs (u, v) for which $u^2 - 2v^2 = \pm 1$ and the integers n for which the sum of the positive integers from 1 till n is a perfect square (namely: if v is odd, then $n = u^2$, and if v is even, then $n = 2v^2$), we see that the integers n for which the sum of the positive integers from 1 till n is a perfect square are the following:

$$1, \quad 8, \quad 49, \quad 288, \quad 1681, \quad 9800, \ldots.$$

Moreover, this is a complete list; no solution is missed out.

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