The Generalised Pythagoras Theorem – Another Proof

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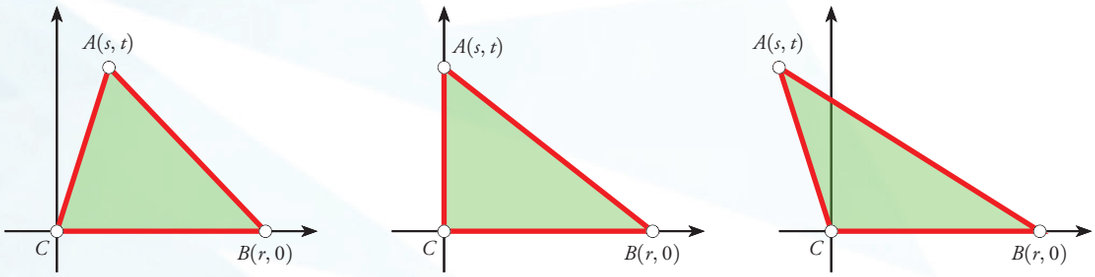
In this short note, we present a proof of the generalised Pythagoras theorem. We use the 'ordinary' Pythagoras theorem for the proof.

Theorem. *In any triangle ABC, we have:*

$$AC^2 + BC^2 > AB^2 \iff \angle C < 90^\circ,$$
$$AC^2 + BC^2 < AB^2 \iff \angle C > 90^\circ.$$

Proof. On the coordinate plane, place the triangle ABC so that vertex C lies at the origin, side CB lies along the positive x-axis, and vertex A lies in the upper half plane (i.e., in the first or the second quadrants); see Figure 1. Let the coordinates of the three vertices be as follows: $C = (0, 0)$; $B = (r, 0)$, where $r > 0$; and $A = (s, t)$, where $t > 0$.



$\angle C < 90^\circ,\ s > 0$ $\quad$ $\angle C = 90^\circ,\ s = 0$ $\quad$ $\angle C > 90^\circ,\ s < 0$

Figure 1.

Then we have: $BC^2 = r^2$, and $AC^2 = s^2 + t^2$, so

$$BC^2 + AC^2 = r^2 + s^2 + t^2.$$

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Also:

$$AB^2 = (r-s)^2 + t^2 = r^2 + s^2 + t^2 - 2rs,$$

so

$$BC^2 + AC^2 - AB^2 = 2rs.$$

Now it is clear from Figure 1 that:

$$\angle C < 90^\circ \iff s > 0,$$
$$\angle C > 90^\circ \iff s < 0.$$

Note that $2rs$ has the same sign as s (since $r > 0$). It follows that

$$BC^2 + AC^2 > AB^2 \iff \angle C < 90^\circ,$$
$$BC^2 + AC^2 < AB^2 \iff \angle C > 90^\circ.$$

We may thus state the "generalised Pythagoras theorem" as follows.

Theorem (Generalised Pythagoras theorem)**.** *In any triangle ABC, we have:*

$$AC^2 + BC^2 > AB^2 \iff \angle C < 90^\circ,$$
$$AC^2 + BC^2 = AB^2 \iff \angle C = 90^\circ,$$
$$AC^2 + BC^2 < AB^2 \iff \angle C > 90^\circ.$$

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