A Note on Geometric Construction

A. K. MALLIK

The whole of plane geometry is based on two figures, the straight line and the circle. Both these figures are defined by two points, say A and B. For drawing these figures, two instruments are available: (i) an unmarked straight edge for drawing a straight line joining A and B and, if necessary, extending the straight line beyond the segment AB on both sides; (ii) a compass for drawing a circle with one of the points A (or B) as centre and passing through the other point B (or A). Attention is drawn to the fact that in Euclid's original text, the compass is regarded as "collapsible." This implies that both ends of the compass—the needle and the pencil—must always be in contact with the drawing plane. The compass 'collapses' as soon as one of the ends is lifted. This, in turn, means that we cannot transfer distances by using the compass or divider in the manner routinely used in schools. It is necessary to emphasize that neither the straight edge nor the compass can be used for measurement. As Borovik and Gardiner [1] say: "Measuring is an *approximate* physical action, rather than an *exact* "mental construction," and so is not really part of mathematics." The emphasis is necessary especially in view of our school geometric box instrument set which consists of a *marked ruler* and a *divider*. The job of the divider, as said earlier, is routinely carried out in school by using the "real compass" which does not collapse like Euclid's compass. Such a compass is generally referred to as a "rusty compass." Note that the use of these instruments for measurement is perfectly acceptable for engineering drawings.

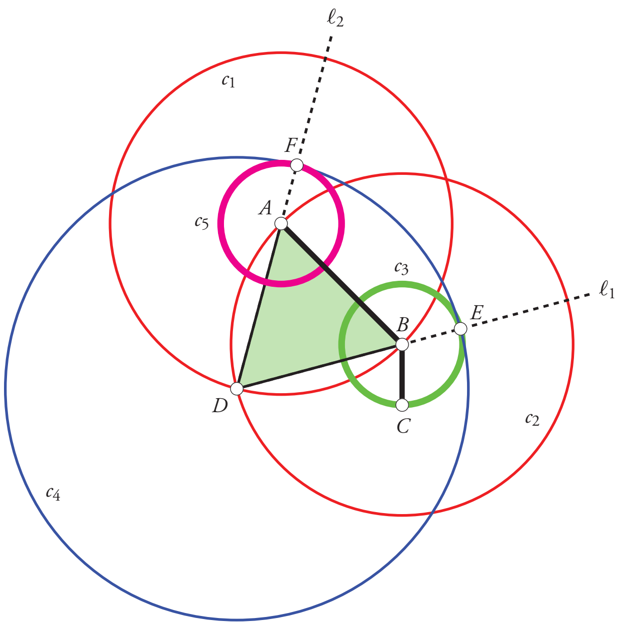
It should be emphasised that the basic figures (straight line and circle) can be drawn only after the two points defining them have been identified. A point is located at the intersection of two straight lines, the intersection of two circles, or the intersection of a straight line with a circle. In the last two cases, two points of intersection are normally generated, unless the two figures touch each other at a single point.

Keywords: Euclidean geometry, constructions, collapsible compass, rusty compass

It is known that any construction achievable by a rusty compass can also be carried out by a collapsible compass. Following [1], we reproduce below one of the construction problems discussed by Euclid to demonstrate how the use of rusty compass is avoided.

Problem. Given three points A, B, C, locate *geometrically* a point F so that line segments AF and BC are equal in length. (Equivalently: draw a circle with centre A and radius equal to the distance between B and C.)

Solution. A common mistake is to draw an arbitrary straight line through A, and then use a rusty compass to locate point F on that line. Two mistakes are thus committed. First, a line is drawn (through A) without locating two points, and then a rusty compass is used for measurement. The correct solution, from [3], is explained in Figure 1.



Steps of the construction

- $c_1 = \text{Circle}\,(A, B)$
- $c_2 = \text{Circle}\,(B, A)$
- $D = c_1 \cap c_2$
- $c_3 = \text{Circle}\,(B, C)$
- $\ell_1 = \text{Ray}\,(D, B)$
- $E = \ell_1 \cap c_3$
- $c_4 = \text{Circle}\,(D, E)$
- $\ell_2 = \text{Ray}\,(D, A)$
- $F = \ell_2 \cap c_4$
- $c_5 = \text{Circle}\,(A, F)$

Figure 1. Notation: Circle (A, B) means: ``Draw a circle with centre $A$, passing through $B$.'' Ray $(A, B)$ means: ``Draw a ray with initial point A, passing through B.'' The steps of the construction are given on the right side. (In the third step, D represents either point of intersection of c_1, c_2.)

(Note: We only consider the case when $AB > BC$. If $AB < BC$, then only small changes are needed in the procedure described in Figure 1. When $AB = BC$, the solution is trivial.)

Discussion of an interesting geometric construction problem

In [2], Jaleel Radhu, a class 11 student, discusses the following construction problem: *Given any $\triangle ABC$, show that there exist points P on side AB and Q on side AC such that $BP = PQ = QC$, and find a way of locating such a pair of points.* The steps in Jaleel's construction are shown in Figure 2.

(1) Locate any point D on side AB such that $BD < AB$ and $BD < AC$.

(2) Through D, draw a line ℓ parallel to side BC.

(3) Locate a point Q_1 on side AC such that $CQ_1 = BD$.

(4) Locate a point P_1 on ℓ such that $Q_1P_1 = BD$.

(5) Through P_1, draw a line parallel to AB; let it intersect side BC at E.

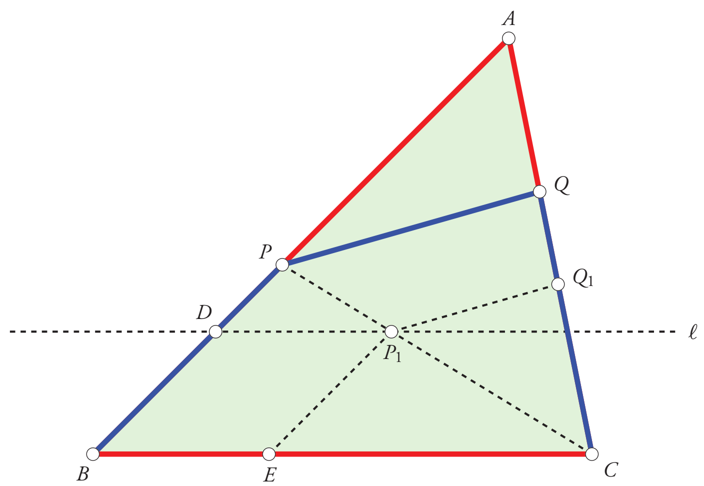


Figure 2.

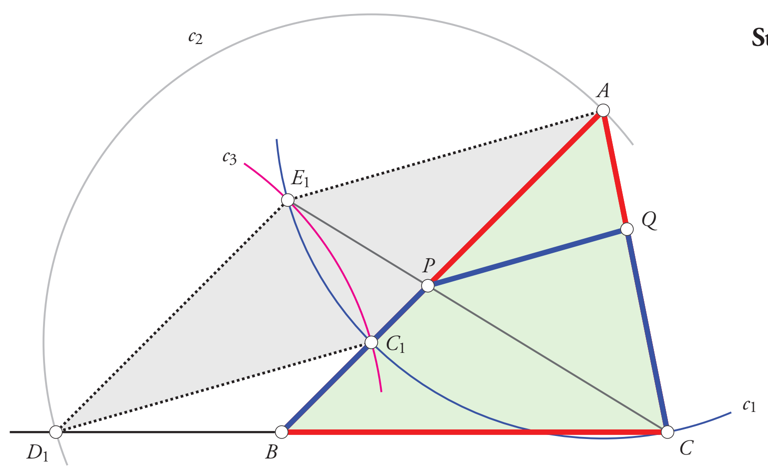
(6) Quadrilateral P_1DBE is a parallelogram; so $P_1E = DB$. Hence $EP_1 = P_1Q_1 = Q_1C$.

(7) Extend CP_1 beyond P_1; let it intersect AB at P. Through P, draw a line parallel to P_1Q_1; let it intersect AC at Q.

(8) Then P and Q are the required pair of points.

Comment on the solution. Note that the point D is located arbitrarily, neither defined in the problem, nor obtained by intersection of lines or circles. As will be shown below, the problem can be solved without the use of an arbitrary point.

Another solution to the problem. Here are the steps of the construction [4]. The diagram has been drawn assuming that $AC < AB$ (Figure 3).



Steps of the construction

- $c_1 = \text{Circle}\,(A, C)$
- $C_1 = c_1 \cap AB$
- $c_2 = \text{Circle}\,(C_1, A)$
- $D_1 = c_2 \cap \text{Ray}\,(C, B)$
- $c_3 = \text{Circle}\,(D_1, C_1)$
- $E_1 = c_1 \cap c_3$
- $P = CE_1 \cap AB$
- $PQ \parallel E_1A$

Figure 3. Quadrilateral $AC_1D_1E_1$ is a rhombus (each side equal to AC). The required points are P and Q.

Proof of construction. Consideration of the similar triangles $\triangle CPB$ and $\triangle CE_1D_1$ yields

$$\frac{CP}{CE_1} = \frac{BP}{D_1E_1}. \qquad (1)$$

Consideration of the similar triangles $\triangle CPQ$ and $\triangle CE_1A$ yields

$$\frac{CP}{CE_1} = \frac{PQ}{E_1A} = \frac{CQ}{CA}. \tag{2}$$

From (1) and (2), we get

$$\frac{BP}{D_1E_1} = \frac{PQ}{E_1A} = \frac{CQ}{CA}.$$

Since by construction $D_1E_1 = AE_1 = AC$, we get $BP = PQ = QC$, as desired.

The rhombus $AC_1D_1E_1$ here lies outside triangle ABC. This is so because $AC > AB/2$.

If $AC < AB/2$, then rhombus $AC_1D_1E_1$ lies inside triangle ABC, and point Q lies on the extension of the side CA, as shown in Figure 4. (Note that if $AC < AB/2$, then, in order to satisfy the triangle inequality $AC + BC > AB$, we must have $BC > AB/2$. Now, following the same procedure, we may obtain points R on side BC and S on side AB such that $CR = RS = AS$.) Finally, if $AC = AB/2$, then P is the midpoint of AB, and Q coincides with A, resulting in $BP = PA = CA = AB/2$.

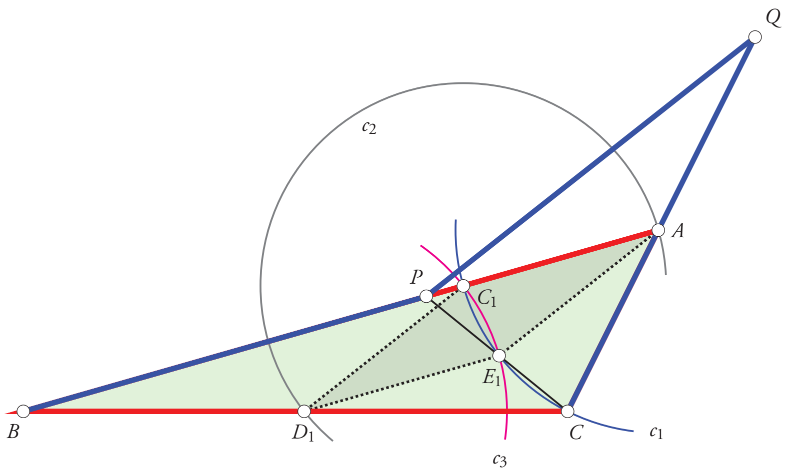


Figure 4. The construction steps are the same as earlier. Quadrilateral $AC_1D_1E_1$ is a rhombus (each side equal to AC). The required points are P and Q.

Acknowledgement. The author wishes to acknowledge the contribution of the editorial board towards improvement of this article and for adding the reference [3].

References

1. Alexandre Borovik and Tony Gardiner, "The Essence of Mathematics Through Elementary Problems", Cambridge , U.K., Open Book Publishers, 2019.
2. Jaleel Radhu, "A Challenging Geometric Construction problem", *At Right Angles*, November 2019, pp. 81–82, https://azimpremjiuniversity.edu.in/SitePages/resources-ara-vol-8-no-5-november-2019-geometric-construction-problem.aspx.
3. Wikipedia, "Compass equivalence theorem", https://en.wikipedia.org/wiki/Compass_equivalence_theorem
4. Asok Kumar Mallik, "Popular Problems and Puzzles in Mathematics", Bangalore, India, IISc Press, 2018, Problem No. 163.

ASOK KUMAR MALLIK is a retired Professor of Mechanical Engineering at Indian Institute of Technology, Kanpur. He has a deep interest in high school mathematics. Besides writing books on Mechanical Engineering subjects, he has authored books on Mathematics at the high school level. He has delivered many popular lectures on Mathematics arranged by institutions in India and abroad. He can be contacted at asokiitk@gmail.com.