Middle School Problems Requiring Exhaustive Search

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There are problems where one is required to find examples or instances that satisfy some given conditions, from many possibilities. One would have to search among several possibilities taking care not to leave out any possibility. Here are a few such problems.

Problem 1. We generally record dates in the format DD-MM-YY, i.e., the day followed by the month and the last two digits of the year. In some cases you may find that (DD)(MM) = (YY), i.e., the product of the date and month equals the (last two digits of the) year. An example is 15-4-60. Find all such dates in a century and present them in chronological order.

Problem 2. In an article on magic squares in the July 2014 issue of AtRiA, the following was stated: 'There are 86 ways of picking out four numbers from the set 1-16 to give a sum of 34.' Find all these combinations.

Problem 3. What is the least natural number that has exactly 100 factors? (Here we include all the factors of the number, not just the prime factors.)

Problem 4. Observe the factorisations below:

$$x^2 + 10x + 24 = (x + 4)(x + 6)$$

$$x^2 + 10x - 24 = (x + 12)(x - 2)$$

The expressions on the left differ only in the sign of the constant term. They are both factorisable. Find all such instances with the constant term under 100, i.e., values of p and q, $q < 100$ that make both $x^2 + px + q$ and $x^2 + px - q$ factorisable.

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Pedagogical Notes: The value of the skill of searching is so obvious (even to students) that one may not need to emphasise this. However, the process during which a random search becomes systematic is worth observing and developing in students. Not only do students learn to reflect on their thinking, but they also begin to analyse and select more efficient ways of conducting their searches. It is important that they document this process and talk about it, rather than just focus on the answers that they get.

Some Hints/Pointers

As these problems are somewhat time consuming, we provide some tips below to ensure that you are on the right track. You may decide not to read this part and proceed on your own. It may be good to work on these as a group.

Hints for problem 1

This is essentially an exercise in expressing the numbers 1 to 99 as products of two numbers in as many ways as possible. Among all such factorisations, you need to pick the ones permissible in view of the restrictions on the dates and month orders.

Hints for problem 2

To start with, we make a ground rule: we present the numbers in any combination in ascending order; this helps avoid repetition.

Start with the lowest combination, (1, 2, 3, 4), which, of course, is not admissible as the sum is less than 34. Keep moving up, first, by increasing the last number, then the third number, followed by the second number and finally, the first number, till you hit the ceiling, i.e., with any further increase the total would exceed 34.

Hints for problem 3

To address this question, you should be aware of the following rule from number theory: "If $N = a^p \times b^q \times c^r \ldots$ is the prime factorisation of a natural number N, where $a, b, c, \ldots$ are prime numbers and $p, q, r, \ldots$ are natural numbers, then the number of factors of N is given by $(p + 1)(q + 1)(r + 1)\ldots$" This includes the factors 1 and the number N itself.

Now, factorise 100 in as many ways as possible. In each factorisation, reduce each factor by 1 and assign these as exponents of suitable prime numbers with a view to keep the products as low as possible. For example, starting with the factorisation $100 = 4 \times 5 \times 5$, we reduce each factor by 1, obtaining the numbers 3, 4, 4. Using the rule stated above, we see that each of the following numbers has exactly 100 factors:

$$2^3 \times 3^4 \times 5^4,\ 2^4 \times 3^3 \times 5^4,\ 2^4 \times 3^4 \times 5^3.$$

These are not the only possibilities; there are several others. Now, from among all these possibilities, we need to choose the smallest.

Hints for problem 4

The factorisation exercise you carried out for Problem 1 should aid you now. Examine the factors of each number from 1 to 99 and look for instances where the sum of two complementary factors equals the difference of two complementary factors. In other words, values that satisfy the equations:

$$A \times B = C \times D = N < 100,$$

$$A + B = C - D.$$

We hope that these exhaustive searches do not turn out to be exhausting searches!

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