Pretty Power Play

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A few months back, while browsing through the pages of LinkedIn (I am, unfortunately, unable to recall the URL), I came across the following set of striking equalities:

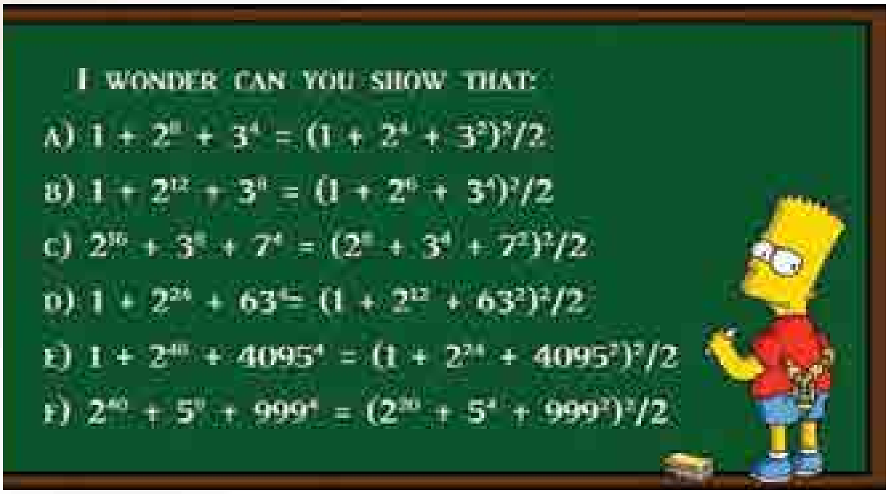

Figure 1. Unusual power equalities

A most remarkable set of equalities! What could be the general law behind them? (There clearly does seem to be some kind of general result hiding behind these separate instances.) Let us see if we can uncover the secret.

We observe that each identity is of the following form, for some positive integers a, b, c and some positive integers k, m, n:

$$a^{2k} + b^{2m} + c^{2n} = \frac{1}{2}\left(a^k + b^m + c^n\right)^2.$$

Moreover, a, b, c are clearly connected through some relation.

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Observe further that in all the cases shown, the exponents on the left side (i.e., $2k, 2m, 2n$) are not just even numbers but are multiples of 4.

This being the case, perhaps it will be easier to spot a pattern if we rewrite the identities in a more uniform manner, so that each term on the left side is a fourth power (i.e., of the form k^4), and each term on the right side is a squared quantity (i.e., of the form k^2). Accordingly, we rewrite the six identities as shown below.

$$\begin{aligned}
1^4 + 4^4 + 3^4 &= \frac{1}{2}\left(1^2 + 4^2 + 3^2\right)^2, \\
1^4 + 8^4 + 9^4 &= \frac{1}{2}\left(1^2 + 8^2 + 9^2\right)^2, \\
16^4 + 9^4 + 7^4 &= \frac{1}{2}\left(16^2 + 9^2 + 7^2\right)^2, \\
1^4 + 64^4 + 63^4 &= \frac{1}{2}\left(1^2 + 64^2 + 63^2\right)^2, \\
1^4 + 4096^4 + 4095^4 &= \frac{1}{2}\left(1^2 + 4096^2 + 4095^2\right)^2, \\
1024^4 + 25^4 + 999^4 &= \frac{1}{2}\left(1024^2 + 25^2 + 999^2\right)^2.
\end{aligned}$$

Each identity now has the following form:

$$a^4 + b^4 + c^4 = \frac{1}{2}\left(a^2 + b^2 + c^2\right)^2. \tag{1}$$

Looking carefully at the numbers involved in the different instances, we quickly notice a pattern connecting a, b, c; namely: in each case, one of the three numbers is equal to the sum of the other two numbers; please check! We have clearly made progress.

To clinch the issue, let us now approach the problem from the opposite end; let us find out what relation must exist between the quantities a, b, c in order that relation (1) reduces to an identity. Here is what we obtain:

$$\begin{aligned}
a^4 + b^4 + c^4 = \frac{1}{2}\left(a^2 + b^2 + c^2\right)^2 &\iff 2\left(a^4 + b^4 + c^4\right) - \left(a^2 + b^2 + c^2\right)^2 = 0 \\
&\iff a^4 - 2a^2(b^2 + c^2) + b^4 - 2b^2c^2 + c^4 = 0.
\end{aligned}$$

It is not obvious how to factorise the expression $a^4 - 2a^2(b^2 + c^2) + b^4 - 2b^2c^2 + c^4$. To this end, let us write $x = a^2$ and treat the last equality as a quadratic equation in x. The resulting equation is easy to solve, as the discriminant turns out to be a perfect square ($\Delta = 4\left((b^2 + c^2)^2 - (b^2 - c^2)^2\right) = 16b^2c^2$). Here is what we get (we have skipped the in-between steps):

$$\begin{aligned}
&x^2 - 2x(b^2 + c^2) + b^4 - 2b^2c^2 + c^4 = 0, \\
&\implies x = (b - c)^2 \text{ or } x = (b + c)^2.
\end{aligned}$$

And now, using the ever-versatile difference-of-two-squares factorisation formula,

$$\begin{aligned}
&a^4 - 2a^2(b^2 + c^2) + b^4 - 2b^2c^2 + c^4 \\
&= \left(a^2 - (b - c)^2\right) \cdot \left(a^2 - (b + c)^2\right) \\
&= (a - b + c) \cdot (a + b - c) \cdot (a - b - c) \cdot (a + b + c).
\end{aligned}$$

It follows that

$$\begin{aligned}
&a^4 + b^4 + c^4 = \frac{1}{2}\left(a^2 + b^2 + c^2\right)^2 \\
&\iff (a - b + c) \cdot (a + b - c) \cdot (a - b - c) \cdot (a + b + c) = 0.
\end{aligned}$$

Well: the secret is now fully revealed! Namely, identity (1) holds if and only if any of the following conditions holds:

$$\begin{aligned} a-b+c&=0,\\ a+b-c&=0,\\ a-b-c&=0,\\ a+b+c&=0. \end{aligned} \tag{2}$$

The first three possibilities may be described compactly by stating that one of the numbers a, b, c equals the sum of the other two numbers.

The remaining possibility (in which the sum of the three numbers is 0) may create the impression of a new relation under which the identity holds, unnoticed earlier; but a closer look reveals that this is not so. For, all the exponents occurring in relation (1) are even numbers, implying that we can freely change the signs of any of the numbers a, b, c, without equality being affected. And if we change the sign of any one of the numbers, we are back in one of the first three possibilities.

Using (1) as a base, arithmetical relations of interest can be deduced; for example:

$$1^2+16^2+25^2=\frac{1}{2}\left(1+16+25\right)^2.$$

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