Geometrical Proof of an Application of Ptolemy's Theorem

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Introduction. A recent article [1] discussed Ptolemy's theorem and applications of the theorem. The author noted that for the first application, there was also a trigonometric solution based on the identity $\sin(60^\circ - \theta) + \sin\theta = \sin(60^\circ + \theta)$. In this note, we present a simple and elegant geometrical proof for this theorem.

Theorem. *Let ABC be an equilateral triangle, and let P be any point on the minor arc BC of its circumcircle. Then* $PA = PB + PC$.

A geometrical proof. Figure 1 depicts the situation. On BP as base, draw an equilateral triangle EBP, with E on the same side of BP as A. Join AE.

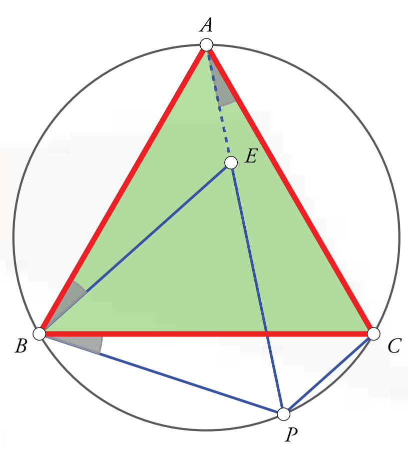

Figure 1. Construction: Triangle *EBP* is equilateral. Join *AE*.

Observe that we have shown AE using a dashed line and EP using a solid line. This is to ensure that we do not unconsciously assume that points A, E, P lie in a straight line.

Keywords: Ptolemy's theorem, equilateral triangle, circum-circle, point, relationship

Since $\measuredangle ABC = \measuredangle EBP$, both being 60°, it follows that $\measuredangle ABE = \measuredangle CBP$ (both angles are marked).

Consider $\triangle ABE$ and $\triangle CBP$. They are congruent to each other (side-angle-side or SAS congruence), therefore $\measuredangle BAE = \measuredangle BCP$. As we also have $\measuredangle BAP = \measuredangle BCP$ ("angles in the same segment"), it follows that $\measuredangle BAE = \measuredangle BAP$, and hence that points A, E, P are collinear. Hence $PA = PE + EA$.

But $PE = PB$, since $\triangle EBP$ is equilateral, and $EA = PC$, by the triangle congruence just proved.

Hence $PA = PB + PC$. ■

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References

1. Shailesh Shirali, "How to Prove it: Ptolemy's Theorem" from *At Right Angles*, Vol.5, No. 3, Nov. 2016. Pp 53-57. http://www.teachersofindia.org/en/ebook/how-prove-it-ptolemys-theorem

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PRIME NUMBER RELATIONS

1. (5 – 3) = (2 × 1)
2. (7 – 5) = (3 – 2) + 1
3. (11 – 7) × (3 – 2) = (5 – 1)
4. (7 – 5) + (3 – 2) = (13 – 11) + 1
5. (11 – 7) × [(5 – 3) – (2 – 1)] = (17 – 13)
6. (7 – 5) × [(3 – 2) + 1] = (19 – 17) + (13 – 11)
7. (11 – 7) × (5 – 3) × (2 – 1) = (17 – 13) + (23 – 19)
8. (29 – 23) + 1 = (19 – 17) + (13 – 11) + (7 – 5) + (3 – 2)
9. [(31 – 29) + (11 – 7) + (5 – 3)] × (2 – 1) = (23 – 19) + (17 – 13)
10. (37 – 31) + (19 – 17) + (7 – 5) = (29 – 23) + (13 – 11) + (3 – 2) + 1
11. (41 – 37) × [(23 – 19) – (31 – 29)] = {(17 – 13)[(11 – 7) – (5 – 3)]} × (2 – 1)
12. (37 – 31) + (43 – 41) + (13 – 11) + 1 = (29 – 23) + (19 – 17) + (7 – 5) + (3 – 2)
13. (47 – 43) + (41 – 37) + (31 – 29) + (5 – 3) = [(23 – 19) + (17 – 13) + (11 – 7)] × (2 – 1)
14. (53 – 47) + (29 – 23) + (7 – 5) = (37 – 31) + (43 – 41) + (19 – 17) + (13 – 11) + (3 – 2) + 1
15. (47 – 43) + (31 – 29) + (23 – 19) + (11 – 7) = [(59 – 53) + (41 – 37) + (17 – 13)] × [(5 – 3) – (2 – 1)]
16. [(61 – 59) + (53 – 47) + (29 – 23)] × 1 = [(43 – 41) + (37 – 31) + (19 – 17) + (13 – 11) + (7 – 5)] × (3 – 2)
17. (67 – 61) + (47 – 43) + (23 – 19) + (11 – 7) = [(59 – 53) + (41 – 37) + (17 – 13) + (31 – 29) + (5 – 3)] × (2 – 1)
18. (53 – 47) + (29 – 23) + (19 – 17) + (7 – 5) + 1 = (71 – 67) + (61 – 59) + (37 – 31) + (43 – 41) + (13 – 11) + (3 – 2)