ClassRoom

Justification of 90° Angle Construction Procedure

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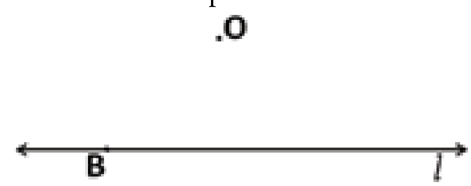
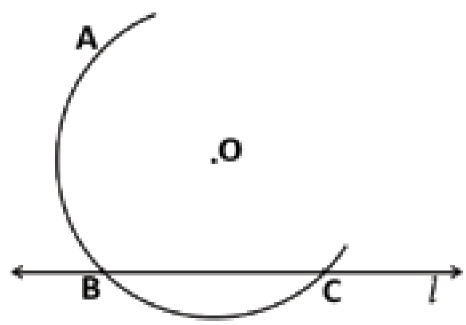
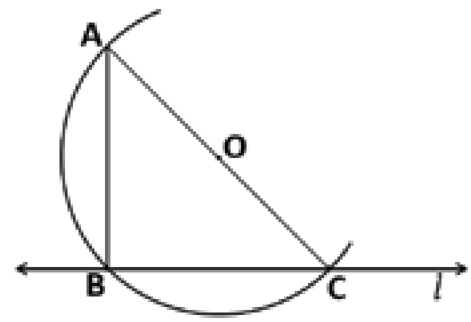
Geometrical construction is one of the interesting topics in geometry and most of us like constructing geometrical figures using compass & ruler. We follow certain processes to complete the construction of the geometrical figures and this process follows certain logical arguments based on properties of shapes and the relation between different geometrical shapes. However, most of our experience reflects two major challenges in the construction of geometrical figures. The first one is following certain processes without understanding the justification of the processes and the second one is not exploring different ways of constructing the same geometrical figure. Therefore, it is necessary to take care of these two major challenges during our classroom teaching.

If we will recall our curriculum, generally we start compass and ruler geometrical construction from class 6. The deductive or axiomatic reasoning behind some geometrical constructions might be difficult to grasp for children at that level so we should use verification to make the children understand the justification at that level.

In this article, we will explore two processes of constructing a right angle and their justification.

Keywords: Geometric construction, angles, justification, alternatives

Process 1 (Construction of 90° angle)

Step 1: Draw a line l and mark a point B on it.  Figure 1	**Step 2:** Mark another point O outside the line l.  Figure 2
Step 3: Place the point of the compass at O and draw an arc that passes through B. Let it intersect the line l again at point C.  Figure 3	**Step 4:** Draw the line segment from point C through O and let it meet the arc at point A. Join AB.  Figure 4

Note: Here we have used the word arc as part of the circumference of a circle.

So, what is the measure of the angle ABC?

The question is whether it is 90°. If so, how do we justify it?

One way of justification could be by measuring the angle using a protractor, based on the level of understanding of the children. Note that using the protractor to measure the angle may give only an approximate value. However, how will we justify it logically?

Proof 1: Given l is a line. A, B and C are three distinct points on the circle γ_1 with centre at O. AC is a diameter of the circle. We have to prove that $\measuredangle ABC$ is a right angle.

Join BO. Here OA, OB and OC are radii of the circle γ_1.

In the triangle AOB, AO = BO,

Hence, triangle AOB is an isosceles triangle.

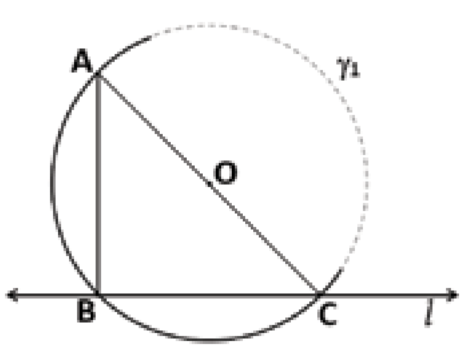


Figure 5

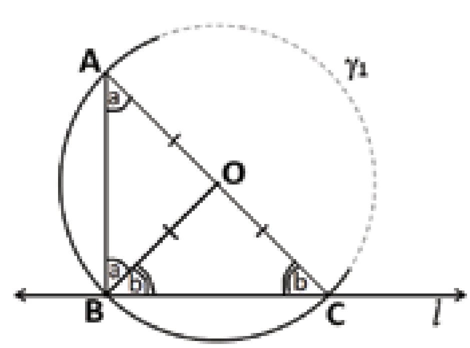


Figure 6

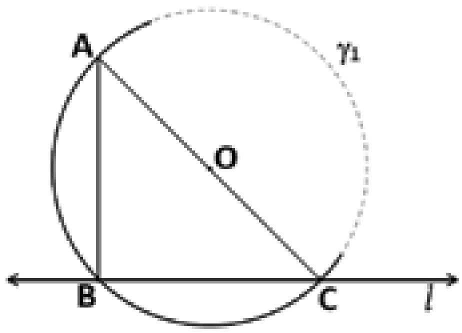


Figure 7

So, ∡OAB = ∡ABO (angles opposite to equal sides of an isosceles triangle are equal).

Let ∡OAB = ∡ABO = a

Similarly, BOC is an isosceles triangle as BO = CO,

So, ∡OBC = ∡BCO (angles opposite to equal sides of an isosceles triangle are equal).

Let ∡OBC = ∡BCO = b

Since ABC is a triangle, the sum of the interior angles will be equal to two right angles.

∡CAB + ∡ABC + ∡BCA = 180°

∡CAB + (∡ABO + ∡OBC) + ∡BCA = 180° (as ∡ABC = ∡ABO + ∡OBC)

$$a + a + b + b = 180^\circ$$
$$2a + 2b = 180^\circ$$
$$2(a + b) = 180^\circ$$
$$a + b = 90^\circ$$
i.e., ∡ABC = 90°

So, we have proved that ∡ABC is a right angle.

Proof 2: Given that l is a line. A, B and C are three distinct points on the circle γ_1 with centre at O. AC is a diameter of the circle. We have to prove that ∡ABC is a right angle.

Here, figure 7 is rotated to Figure 8 to visualize the coordinates easily.

Let the co-ordinates of O be (0,0), r be the radius of the circle and CA be on the X-axis (as in figure 8); so the co-ordinates of A are (r, 0) and the coordinates of C are (-r, 0). B is a point on the circle of radius r and OB makes an angle θ with

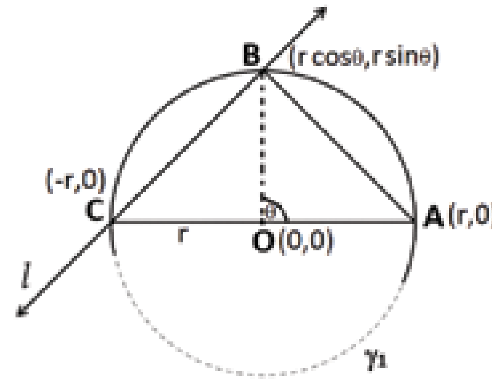


Figure 8

the positive X-axis. Using the polar coordinate system, the coordinates of point B will be ($r\cos\theta$, $r\sin\theta$). We can also find the coordinates of points A and C using the polar coordinate system by applying θ as 0° for point A and 180° for point B in the coordinates ($r\cos\theta$, $r\sin\theta$).

Now the slope of AB $= \dfrac{r\sin\theta}{r\cos\theta - r}$

The slope of BC $= \dfrac{r\sin\theta}{r\cos\theta + r}$

To prove ∡ABC is a right angle, one way could be to show that AB is perpendicular to BC. This is possible, if one could show that the product of slopes of both the lines is −1.

$$\text{(Slope of AB) . (Slope of BC)}$$
$$= \frac{r\sin\theta}{r\cos\theta - r} \cdot \frac{r\sin\theta}{r\cos\theta + r}$$
$$= \frac{r^2\ \sin^2\theta}{r^2\ \cos^2\theta - r^2}$$
$$= \frac{r^2\ \sin^2\theta}{r^2\ (cos^2\theta - 1)}$$
$$= \frac{\sin^2\theta}{\cos^2\theta - 1}$$
$$= \frac{\sin^2\theta}{-\sin^2\theta} = -1$$

So, AB is perpendicular to BC. Hence ∡ABC is a right angle.

This construction is based on Thales's theorem which states: If A, B, and C are distinct points on a circle in which the line segment AC is a diameter, then ∡ABC is a right angle. Or generally we say that the angle in a semi-circle is a right angle.

Process 2 (Construction of 90° angle)
This process of constructing a right angle (90°) is common in most of our state and NCERT textbooks. This construction includes the two concepts (as in Figure 9) – construction of an angle of 60° (or multiple of 60°) and bisector of an angle. Here it is the bisector of 60°. As in Figure 9, $\measuredangle CAB = 60°$ and $\measuredangle EAC = 30°$ EA being the bisector of angle $\measuredangle DAC$.

$$\text{So, } \measuredangle EAB = \measuredangle EAC + \measuredangle CAB$$
$$= 30° + 60° = 90°.$$

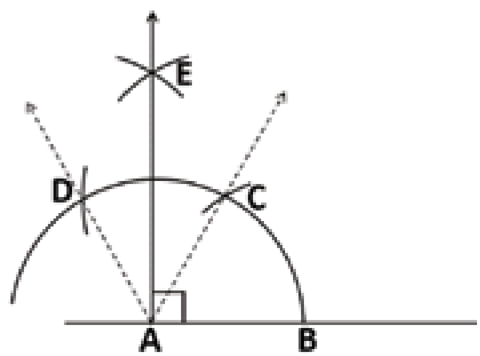


Figure 9

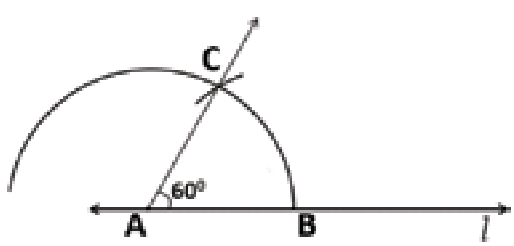


Figure 10

To justify that $\measuredangle EAB$ is 90°, we have to justify (i) the construction of 60° angle and (ii) the bisection of an angle.

Construction of angle 60°
Step 1: Draw a line l and mark a point A on it.

Step 2: Place the point of the compass at A and draw an arc of convenient radius that cuts the line l at a point B.

Step 3: Keeping the width unchanged, place the point at B and draw an arc that cuts the previous arc at C.

Step 4: Draw the ray AC.

Here angle CAB is 60°.

Justification: We have to prove that $\measuredangle CAB$ is 60°. Join CB as in figure 11. In the steps of the construction of an angle of 60°, we do not change the radius of the arc. So, the circles γ_1 and γ_2 have the same radii and the centres of both circles lie on the endpoints of AB, and AB is the radius of both the circles.

In circle γ_1, AB = AC as they are radii of the circle,

In circle γ_2, BA = BC as they are radii of the circle,

so, AB = BC = AC.

Hence, the triangle ABC is an equilateral triangle. Since the sum of the interior angles is 180° and all the angles are equal, each angle will be equal to 60°.

Construction of Angle Bisector. Given an angle CAB, suppose we wish to bisect the angle. With B as centre, draw an arc whose radius is more than half the length BC. With the same radius and with C as centre draw another arc. Let the two arcs intersect at D (as in figure 14). Draw the ray AD, from A through D. Now the ray AD bisects the angle CAB.

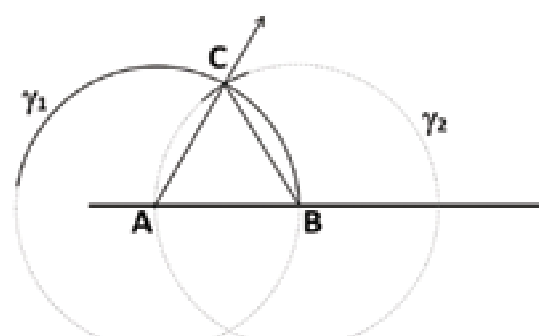


Figure 11

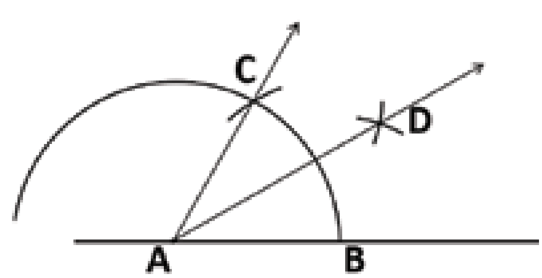


Figure 12

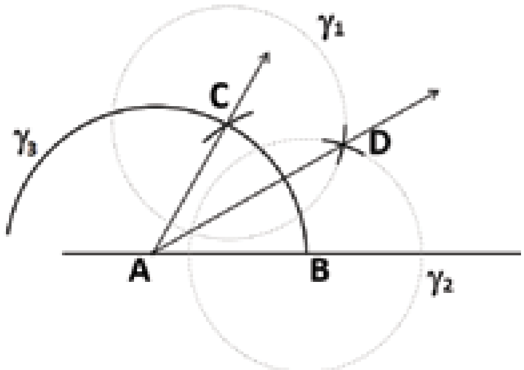


Figure 13

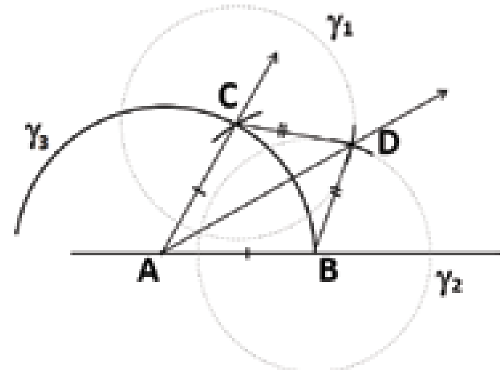


Figure 14

Justification: In figure 14, AB = AC being the radii of circle γ_3. CD = BD as we have chosen the same radius for the arcs $\gamma_2 \& \gamma_1$.

We have to prove that the ray AD bisects the angle CAB,

Now consider the triangle ACD and triangle ABD

AC = AB

CD = BD

AD is common

So, by the side-side-side property, triangle ACD is congruent to triangle ABD.

Hence $\measuredangle$CAD = $\measuredangle$DAB

So, AD is the bisector of the angle $\measuredangle$CAB.

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TWO PROBLEMS

PROBLEM 1

Someone computed the factorial of some large number (recall that if *n* is a natural number, then '*n* factorial' (written *n*!) is the product of all the numbers from 1 till *n*) and obtained the following gargantuan number:

81591528324789773■34■5611269596115894272000000000.

Unfortunately, some blobs of red paint fell on two digits right in the middle (shown as red rectangles). Can you figure out what those two digits could be?

PROBLEM 2

Look at the following beautiful relation which expresses 2019 in terms of the digits 1,2,3,…,9:

$$9 + (8 + 7) \times \left(\frac{6!}{5} - 4 - 3 - 2 - 1\right) = 2019.$$

Can you find a similar expression for 2020?