Low Floor High Ceiling Tasks

Summing V

The Thinking Skills Pullout continues to be our favorite as it packs a lot of food for thought. It has already generated a series of two articles including a LFHC (Dotted Squares) and a TearOut. Here is another LFHC (which we have tried with a group of children from Class 3-4 in Pokhrama, Bihar, and with government school teachers from Telengana).

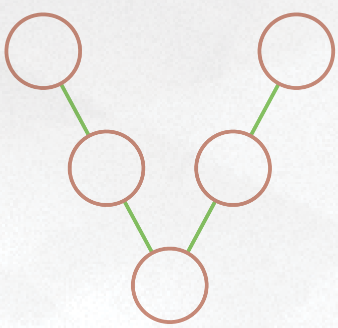

Figure 1

Make a V with 5 circles as in Figure 1. Pick 5 consecutive numbers (e.g. 13, 14, 15, 16, 17).

1. Arrange them on this so that the two arms of the V have the same total.
   a. Which number is at the centre? What is the total of each arm?
   b. Keeping the centre fixed, can you rearrange the remaining numbers in another way to get the same total for both arms? Did the total change?
   c. How many arrangements are possible?

Teacher Note: The 3rd number of the set can be a natural choice for the number at the centre. The remaining 4 numbers can be split into 2 groups with the same totals since the sum of numbers which are equidistant from the centre is constant (for example, with 13 at the centre, we get 11 + 15 = 12 + 14). This can happen in 2 ways. Since the left and the right arms of the V can be switched, the number of ways doubles to 4. Within each group on each arm, there are 2 choices for the number at the end. So, with the same number at the centre, there are 8 ways of choosing the numbers for the arms. For the set {11, 12, 13, 14, 15}, the eight possibilities are:

11 – 15 – 13 – 12 – 14	15 – 11 – 13 – 12 – 14	11 – 15 – 13 – 14 – 12	15 – 11 – 13 – 14 – 12
14 – 12 – 13 – 15 – 11	14 – 12 – 13 – 11 – 15	12 – 14 – 13 – 15 – 11	12 – 14 – 13 – 11 – 15

2. Now find another arrangement with the same 5 numbers but with a different number at the centre.
   a. Which number is at the centre now? Did the total change? By how much?

b. Can any other number be at the centre? What is the total for that?

c. Are there some numbers which cannot be at the centre? Why?

Teacher Note: Out of the 5 consecutive numbers, the first, the third and the fifth can be at the centre, e.g. for 11, 12, 13, 14, 15, the number at the centre can be 11, 13 or 15. The totals change with the centre and are consecutive numbers as well.

The 2^{nd} and the 4^{th} number cannot be in the centre. For any 5 consecutive numbers, this can be checked by trying out all possible combinations. But something more is needed to generalize for any such set. That can be tackled with parity. If we put, say, the 2^{nd} number at the centre, then out of the 4 remaining numbers, 3 are odd and 1 is even (e.g. 11, 13, 15 odd and 14 even) or 3 are even and 1 is odd (e.g. for 8, 9, 10, 11, 12, if 9 is at the centre, 8, 10, 12 are even and 9 is odd). So, if these 4 numbers are split in 2 groups, then the total for one group is odd and for the other is even. So, the two group totals cannot be the same. Therefore, the 2^{nd} number cannot be at the centre. Similarly, we can show that the 4^{th} number also cannot be at the centre.

3. Try the above with other sets of five consecutive numbers.
   a. Record your findings in this table.
   b. Do you see any patterns? What are they?
   c. If the total is 72, can you find out which five numbers are chosen? Which number is at the centre?
   d. Repeat the same if the total is 17. What if the total is 43?

Teacher Note: We observe the following:

i. The totals are consecutive numbers

ii. The middle total i.e. the total when the 3^{rd} number is at the centre is a multiple of 3

iii. In fact, it is thrice the 3^{rd} number

So, the totals are 3×3^{rd} number – 1, 3×3^{rd} number or 3×3^{rd} number + 1

4. Following the pattern:
   a. Suppose the totals are 12, can you predict the 5 consecutive numbers, and which one is at the centre?
   b. Can you do the same if the total is 23? Or 16?
   c. What can we generalize?

Teacher Note: $12 = 3 \times 4$, so, 4 is the number at the centre and is the 3^{rd} of the 5 numbers. Therefore the 5 numbers are 2, 3, 4, 5 and 6.

$23 = 3 \times 8 - 1$, so, 8 is the 3^{rd} of the 5 numbers. Therefore, the numbers are 6, 7, 8, 9 and 10 with 6 at the centre.

Similarly, $16 = 3 \times 5 + 1$, so, 5 is the 3^{rd} of the 5 numbers which are 3, 4, 5, 6 and 7 with 7 at the centre.

If we represent the 5 numbers as $n - 2$, $n - 1$, n, $n + 1$, $n + 2$ then we get the following totals which explains the above. Note, similar deduction is possible with n, $n + 1$, $n + 2$, $n + 3$, $n + 4$ as well.

Centre	Total
$n - 2$	$(n - 2) + (n - 1) + (n + 2) = 3n - 1$
n	$n + (n - 1) + (n + 1) = 3n$
$n + 2$	$(n + 2) + (n - 2) + (n + 1) = 3n + 1$

Set of numbers	Number at the centre	**Total**	Number at the centre	**Total**	Number at the centre	**Total**
9, 10, 11 12, 13	9	**32**	11	**33**	13	**34**

5. Exploring other kinds of numbers:
   a. How would this change if you take 5 consecutive even numbers?
   b. Or 5 consecutive odd numbers?
   c. Can you generalize this further?

Teacher Note: For 5 consecutive even numbers, we can consider $2n - 4$, $2n - 2$, $2n$, $2n + 2$, $2n + 4$ to get the following. For odd, we can consider $2n - 3$, $2n - 1$, $2n + 1$, $2n + 3$, $2n + 5$.

Even		Odd	
Centre	Total	Centre	Total
$2n - 4$	$6n - 2$	$2n - 3$	$6n + 1$
$2n$	$6n$	$2n + 1$	$6n + 3$
$2n + 4$	$6n + 2$	$2n + 5$	$6n + 5$

This can be generalized to 5 consecutive numbers in any arithmetic progression i.e. any 5 numbers of the form $n - 2k$, $n - k$, n, $n + k$, $n + 2k$ for any $n, k = 1, 2, 3, \ldots$. We leave the readers to find out what the totals are going to be.

6. Explore similar possibilities for 7 consecutive numbers in a V with 7 circles as in Figure 2 with the same total for each arm.

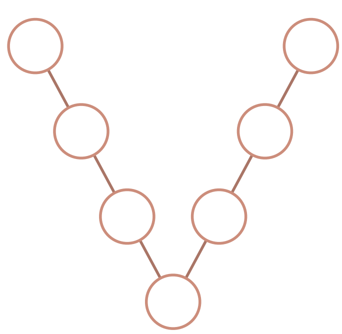

Figure 2

   a. Try with a set starting with an odd number.
   b. Which numbers can be at the centre? Why?
   c. Repeat for a set starting with an even number. Did the pattern change? How?

Teacher Note: The middle most number i.e. the 4th number remains a natural choice. It turns out that the smallest or the biggest numbers can't be at the centre. But the 2nd, 4th and 6th numbers can be. For example, if the numbers are 1, 2, 3, 4, 5, 6, 7, then 2, 4 and 6 can be at the centre. This set has 4 odd numbers and 3 even numbers. Now if an odd number is at the centre, then the remaining 3 odd and 3 even numbers add up to an odd number which can't be halved. So, the centre must be an even number. Similarly, if the numbers are 2, 3, 4, 5, 6, 7, 8, then there are 4 even and 3 odd numbers. So, an odd number can be at the centre and the remaining 4 even and 2 odd numbers can be put in two groups with equal sums. In either case, for 7 circles, the centre can be the 2nd, 4th or the 6th number.

The possible totals are again 3 consecutive numbers with the middle one being 4 times the 4th of the 7 numbers, e.g. totals for 1, 2, 3, 4, 5, 6, 7 are 15, 16 and 17 with 2, 4 and 6 at the centre respectively. Algebraically speaking, if we have $n - 3$, $n - 2$, $n - 1$, n, $n + 1$, $n + 2$, $n + 3$ then the possible centres and corresponding totals are:

Centre	Total
$n - 2$	$(n - 2) + (n - 3) + (n + 3) + (n + 1) = 4n - 1$
n	$n + (n - 3) + (n + 1) + (n + 2) = 4n$
$n + 2$	$(n + 2) + (n - 3) + (n + 3) + (n - 1) = 4n + 1$

So, if we take 7 consecutive even numbers, then the totals would be 3 consecutive even numbers with the middle one being 4 times the 4th even number. We encourage the reader to explore the totals for 7 consecutive odd numbers.

7. What happens for 9 consecutive numbers in a V with 9 circles with same total for each arm?
   a. Which numbers can be at the centre?
   b. What is the total for each centre?
   c. For each centre, how many possible arrangements are there?

Teacher Note: With 9 circles, we get similar pattern as for 5 i.e. 1st, 3rd, 5th, 7th and 9th numbers can be at the centre. The totals turn out to be 5 consecutive numbers with the middle one being 5 times the 5th number.

For each centre, we now get more possible combinations for filling the V. If the centre is the 1st, 5th or 9th number then the remaining 8 numbers form 4 pairs with equal sums. So, there are (4/2) × 2 choices for distributing these 8 numbers in the left and the right groups. Within each group, the numbers can be arranged in 4! ways. So, for each of these centres, there are (4/2) × 2 × $(4!)^2$ ways to fill the V! If the centre is the 3^{rd} or 7^{th} number, then the remaining 8 numbers form 4 pairs, 2 with a smaller sum and 2 with a larger one. So, there are 2 × 2 × 2 = 8 choices for the left and right groups. So, combining the arrangements within each group there are 8 × $(4!)^2$ ways to fill the V.

8. Generalize for a V with $2m + 1$ circles and $2m + 1$ consecutive numbers with same total for each arm.
   a. If m is even, what do you get for the answers to questions 6 *a, b* in terms of $m = 2k$?
   b. If m is odd, what do you get in terms of $m = 2k + 1$?

Teacher Note: In general, the middle most number i.e. the $(m + 1)^{th}$ number can always be at the centre. And every alternate number from the $(m + 1)^{th}$ number can be at the centre. So, if m is even, then the 1st, 3rd, … $(m + 1)^{th}$, … $(2m - 1)^{th}$ and $(2m + 1)^{th}$ numbers can be at the centre i.e. $m + 1$ choices. If m is odd, then 2^{nd}, 4^{th}, … $(m + 1)^{th}$, … $(2m - 2)^{th}$ and $(2m)^{th}$ numbers can be at the centre i.e. m choices. Note that it is $2k + 1$ in both cases.

Let the $(2m + 1)$ consecutive numbers be $n - m$, $n - m + 1, \ldots, n - 1, n, n + 1, \ldots n + m$. So, the number in the centre can be

- $n - m, n - m + 2, \ldots, n - 2, n, n + 2, \ldots, n + m$ for even m
- $n - m + 1, n - m + 2, \ldots, n - 2, n, n + 2, \ldots, n + m - 1$ for odd m

The total with $(m + 1)^{th}$ number i.e. n at the centre is $(m + 1) \times n$. Let us see how this total changes with the number at the centre. The sum of all the $2m + 1$ consecutive numbers $n - m, n - m + 1, \ldots, n - 1, n, n + 1, \ldots n + m$ is $(2m + 1) \times n$. Now, if one of the numbers, say x, is at the centre, then the sum of the remaining $2m$ numbers is $(2m + 1)n - x$. This total gets split equally along each arm of the V. So, each arm gets $\frac{1}{2}[(2m + 1)n - x]$ and x at the centre. So, the total for each arm is $\frac{1}{2}[(2m + 1)n + x]$. The following table lists the centres with the corresponding totals. So, the $2k + 1$ possible totals are also consecutive numbers viz. $(m + 1)n - k, \ldots (m + 1)n - 1, (m + 1)n, \ldots (m + 1)n + k$ for both even and odd m.

Centre x	Total $\frac{1}{2}[(2m + 1)n + x]$
⋮	⋮
$n - 4$	$(m + 1)n - 2$
$n - 2$	$(m + 1)n - 1$
n	$(m + 1)n$
$n + 2$	$(m + 1)n + 1$
$n + 4$	$(m + 1)n + 2$
⋮	⋮

We found this to be an interesting exploration to demonstrate the power of algebra in terms of finding the entire set of 5 numbers and which of them is at the centre just from the totals. Later, by the power of algebra further patterns could be explored and proved. It starts with an exploration with adding numbers (with some constraints) but unfolds to touch the sum of consecutive integers and quite a bit of arithmetic progressions. It can be easily generalized for $2m + 1$ numbers in arithmetic progression on a V with $2m + 1$ circles.

It also involves a combinatorial challenge of how many possible Vs can be made with a given set of numbers. We encourage the brave reader to explore that further for the general case!

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in