ClassRoom

Orthocentre of a Triangle and its Distance from any Point in the Plane

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In this note, we derive a formula that gives the distance between the orthocentre and any arbitrary point in the plane of the triangle. We also discuss some inequalities and other consequences that follow from this relation.

Introduction

The identity described here, which gives the distance between the orthocentre of a triangle and any point in the plane of the triangle, came about when we were discussing the various triangle centres in class: the circumcentre, the incentre, the orthocentre, the centroid and the nine-point centre. We were discussing the formulas for the distances between these centres, but found the topic confusing. Our math mentor challenged us to find a generalised identity in this topic, making use of Stewart's theorem. On taking up this challenge, I came up with the identity described below.

Basic notation

Let ABC be any triangle. We denote its side-lengths by $a = BC$, $b = AC$, $c = AB$, its angles by $\measuredangle A$, $\measuredangle B$, $\measuredangle C$, its semi perimeter by $s = \frac{1}{2}(a + b + c)$, and its area by Δ. Its classical centres are the centre N of the nine-point circle, the circumcentre O, the incentre I, the centroid G, and the orthocentre H. We write I_1, I_2, I_3 for the ex-centres opposite A, B, C, respectively. The classical radii are the circumradius R, inradius r, and exradii r_1, r_2, r_3. (See Figure 1.)

Keywords: Stewart's theorem, orthocentre, inequality

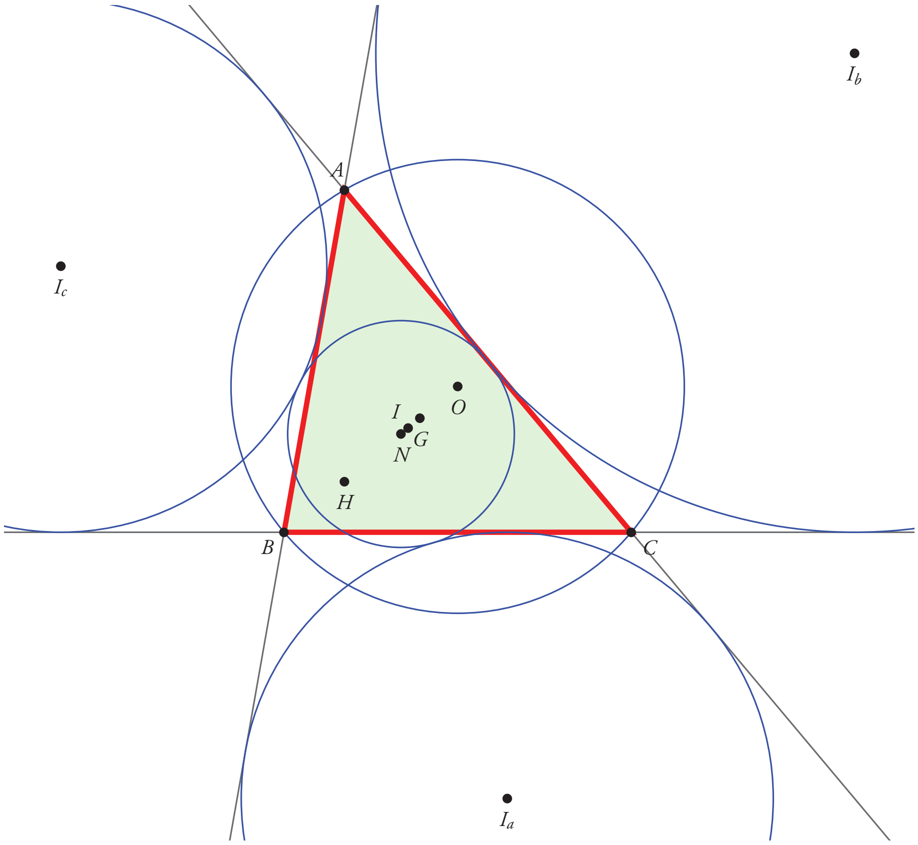

Figure 1.

Let AD, BE, CF be the altitudes of the triangle, concurrent at orthocentre H (Figure 2). The following relations are easily verified using simple trigonometry.

$$BD = c\cos B = 2R\sin C\,\cos B, \qquad CD = b\cos C = 2R\sin B\,\cos C,$$
$$AH = 2R\cos A, \qquad HD = 2R\cos B\,\cos C, \qquad AD = 2R\sin B\,\sin C.$$

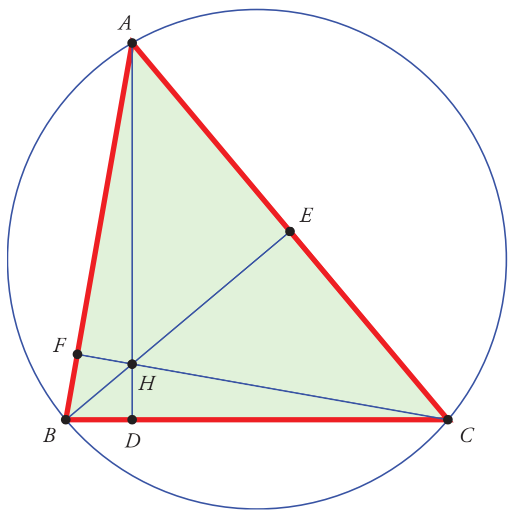

Figure 2.

Connections between different elements of a triangle

We list here a few relations that hold between the elements of a triangle. For the proofs, please refer to the appendix. (Some proofs are given only in outline form.)

Equivalent formulas for the area of a triangle. Four well-known formulas for the area Δ of a triangle (all easily proved):

$$\Delta = rs = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C.$$

Distance from vertex to incentre.

$$AI^2 = \frac{r^2}{\sin^2 A/2}.$$

Similarly for BI^2 and CI^2.

Formulas connecting the lengths of the sides of the triangle.

$$ab + bc + ca = r^2 + s^2 + 4Rr,$$
$$a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right)$$

Formulas connecting the cosines of the angles.

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R},$$
$$\cos A \cos B + \cos B \cos C + \cos C \cos A = \frac{r^2 + s^2 + 4Rr}{4R^2} - \left(1 + \frac{r}{R}\right),$$
$$\cos A \cos B \cos C = \frac{s^2 - r^2 - 4R^2 - 4Rr}{4R^2}.$$

Formulas connecting the sines of the double angles.

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C = \frac{abc}{2R^3} = \frac{2\Delta}{R^2}$$
$$\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C,$$

with similar formulas for $\sin 2A - \sin 2B + \sin 2C$ and $-\sin 2A + \sin 2B + \sin 2C$.

Stewart's theorem

Stewart's theorem gives a relation between the lengths of the sides of a triangle and the length of a cevian of the triangle. (Note. A *cevian* of a triangle is the line segment joining a vertex of the triangle to any point on the side opposite that vertex.) It is named after the Scottish mathematician Matthew Stewart who published the theorem in 1746. See Figure 3 for the notation.

Theorem (Stewart, 1746)**.** *Let a, b, c be the lengths of the sides of a triangle. Let l be the length of a cevian to the side of length a. If this cevian divides a into segments of lengths m and n respectively, with m adjacent to c and n adjacent to b, then*

$$al^2 = mb^2 + nc^2 - amn. \qquad (1)$$

With these preliminary results in place, the main result may be stated as follows.

Theorem 1. *Let M be any point in the plane of an acute triangle ABC with orthocentre H. Then:*

$$HM^2 = \cot B \cot C \cdot AM^2 + \cot C \cot A \cdot BM^2 + \cot A \cot B \cdot CM^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

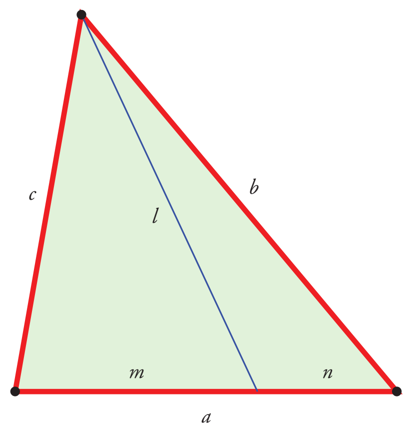

Figure 3.

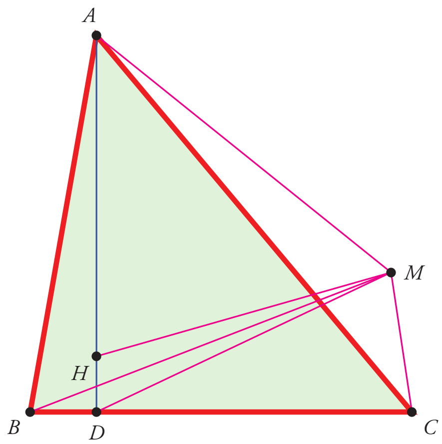

Figure 4.

Proof. We apply Stewart's theorem to triangle MBC in which MD is a cevian. See Figure 4. We get:

$$a \cdot MD^2 = BD \cdot MC^2 + CD \cdot MB^2 - a \cdot BD \cdot DC,$$

$$\therefore\ a \cdot MD^2 = c \cos B \cdot MC^2 + b \cos C \cdot MB^2 - abc \cos B \cos C.$$

Next, we apply Stewart's theorem to triangle MAD in which MH is a cevian. We get:

$$AD \cdot MH^2 = AH \cdot MD^2 + HD \cdot MA^2 - AD \cdot AH \cdot HD,$$

$$\therefore\ MH^2 = \frac{aR\cos A}{\Delta} MD^2 + \cot B \cot C \cdot MA^2 - 4R^2 \cdot \cos A \cos B \cos C.$$

This further gives:

$$\begin{aligned} MH^2 &= \cot A \cot B \cdot MC^2 + \cot C \cot A \cdot MB^2 + \cot B \cot C \cdot MA^2 \\ &\qquad - \frac{abc\, R \cos A \cos B \cos C}{\Delta} - 4R^2 \cdot \cos A \cos B \cos C \\ &= \cot A \cot B \cdot MC^2 + \cot C \cot A \cdot MB^2 + \cot B \cot C \cdot MA^2 - 8R^2 \cdot \cos A \cos B \cos C, \end{aligned}$$

since $abc = 4R\Delta$. Hence:

$$MH^2 = \cot B \cot C \cdot MA^2 + \cot C \cot A \cdot MB^2 + \cot A \cot B \cdot MC^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

This proves the stated identity.

Corollary 1.

$$HM^2 = \cot B \cot C \cdot AM^2 + \cot C \cot A \cdot BM^2 + \cot A \cot B \cdot CM^2 - 4\Delta \cdot \cot A \cot B \cot C.$$

Proof. The relation follows from Theorem 1, by using the identity

$$\Delta = 2R^2 \cdot \sin A \sin B \sin C$$

and making the necessary substitution.

Corollary 2.

$$\begin{aligned} HM^2 = \frac{1}{16\Delta^2} \big[& \left(a^2+b^2-c^2\right)\left(a^2+c^2-b^2\right) AM^2 \\ & + \left(b^2+c^2-a^2\right)\left(b^2+a^2-c^2\right) BM^2 \\ & + \left(c^2+a^2-b^2\right)\left(c^2+b^2-a^2\right) CM^2 \\ & - \left(a^2+c^2-b^2\right)\left(b^2+c^2-a^2\right)\left(b^2+a^2-c^2\right)\big]. \end{aligned}$$

Proof. The relation follows from Corollary 1 by using the following:

$$\cot A = \frac{\cos A}{\sin A} = \frac{\left(b^2+c^2-a^2\right)/(2bc)}{a/(2R)} = \frac{R\left(b^2+c^2-a^2\right)}{abc} = \frac{b^2+c^2-a^2}{4\Delta},$$

with similar relationships for $\cot B$ and $\cot C$. On substituting these into Corollary 1, we obtain the stated result.

Three applications of the identities

Theorem 2.

$$OH^2 = R^2\left(1 - 8\cos A \cos B \cos C\right).$$

Proof. In Theorem 1, replace M by the circumcentre O. We get:

$$OH^2 = \cot B \cot C \cdot AO^2 + \cot C \cot A \cdot BO^2 + \cot A \cot B \cdot CO^2 - 8R^2 \cdot \cos A \cos B \cos C.$$

But $OA = OB = OC = R$, and for any triangle,

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

This implies $OH^2 = R^2\left(1 - 8\cos A \cos B \cos C\right)$, as claimed.

Theorem 3. *For any acute triangle ABC,*

$$\cos A \cos B \cos C \leq \frac{1}{8},$$

with equality when the triangle is equilateral

Proof. This follows from Theorem 2. We know that $OH^2 \geq 0$. Hence it must be that $\cos A \cos B \cos C \leq \frac{1}{8}$.

For equality to hold, we must have $OH = 0$, i.e., the circumcentre coincides with the orthocentre. It may readily be shown that this holds only when the triangle is equilateral. Hence proved.

Theorem 4.

$$IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C.$$

Proof. In Theorem 1, replace M by the incentre I. We get:

$$IH^2 = \cot B\,\cot C \cdot AI^2 + \cot C\,\cot A \cdot BI^2 + \cot A\,\cot B \cdot CI^2 - 8R^2 \cdot \cos A\,\cos B\,\cos C,$$

i.e.,

$$IH^2 = \sum_{A,B,C} \cot B\,\cot C \cdot AI^2 - 8R^2 \cdot \cos A\,\cos B\,\cos C.$$

For any triangle, $\cot B\,\cot C + \cot C\,\cot A + \cot A\,\cot B = 1$ and $AI^2 = bc - 4Rr$. Hence:

$$\begin{aligned}
IH^2 &= \sum_{A,B,C} \cot B\,\cot C \cdot (bc - 4Rr) - 8R^2 \cdot \cos A\,\cos B\,\cos C \\
&= \sum_{A,B,C} (bc\,\cot B\,\cot C) - 4Rr - 8R^2 \cdot \cos A\,\cos B\,\cos C \\
&= \sum_{A,B,C} \left(4R^2 \cos B\,\cos C\right) - 4Rr - 8R^2 \cdot \cos A\,\cos B\,\cos C \\
&= 4R^2 \left[\left(\frac{r^2 + s^2 + 4Rr}{4R^2}\right) - \left(1 + \frac{r}{R}\right)\right] - 4Rr - 8R^2 \cdot \cos A\,\cos B\,\cos C \\
&= r^2 + s^2 - 4R^2 - 4Rr - 8R^2 \cdot \cos A\,\cos B\,\cos C \\
&= r^2 + r^2 + 4R^2 \cdot \cos A\,\cos B\,\cos C - 8R^2 \cdot \cos A\,\cos B\,\cos C \\
&\qquad\qquad \left(\text{since } \cos A\,\cos B\,\cos C = \frac{s^2 - r^2 - 4R^2 - 4Rr}{4R^2}\right),
\end{aligned}$$

$$\therefore\ IH^2 = 2r^2 - 4R^2 \cos A\,\cos B\,\cos C.$$

Hence proved.

We state the following results without proof.

Theorem 5. *Given an acute triangle ABC, the point M in the plane of the triangle which minimises the expression*

$$\cot B\,\cot C \cdot AM^2 + \cot C\,\cot A \cdot BM^2 + \cot A\,\cot B \cdot CM^2$$

is the orthocentre H.

Corollary 3. *For all points M in the plane of the triangle,*

$$\cot B\,\cot C \cdot AM^2 + \cot C\,\cot A \cdot BM^2 + \cot A\,\cot B \cdot CM^2 \geq 8R^2 \cdot \cos A\,\cos B\,\cos C.$$

Theorem 6. *Given an acute triangle ABC and any point M in the plane of the triangle, the following inequalities are true.*

$$\frac{AM^2}{\cot A} + \frac{BM^2}{\cot B} + \frac{CM^2}{\cot C} \geq 4\Delta,$$

$$\frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \geq 1.$$

Moreover, the equality sign holds precisely when M coincides with the orthocentre H.

Theorem 7. *Given an acute triangle ABC, its incentre I, its ex-centres I_1, I_2, I_3 and any point M in the plane of the triangle, we have:*

$$s \cdot IM^2 = (s - a) \cdot I_1M^2 + (s - b) \cdot I_2M^2 + (s - c) \cdot I_3M^2 - 8R\Delta.$$

Conclusion. We have derived a few relations that hold between the elements of a triangle. Theorem 7 is a beautiful result connecting the incentre and excentres. Proceeding in a similar way, we can discover and prove many new relations and inequalities.

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Appendix: Proofs of some of the background results

Formulas connecting the lengths of the sides of the triangle.

$$ab + bc + ca = r^2 + s^2 + 4Rr,$$
$$a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right).$$

Proof. We start with the relation $rs = \sqrt{s(s-a)(s-b)(s-c)}$. Squaring both sides and dividing by s, we get:

$$\begin{aligned} r^2 s &= (s-a)(s-b)(s-c) \\ &= s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc \\ &= s^3 - 2s \cdot s^2 + (ab+bc+ca)s - 4R \cdot rs, \\ \therefore \quad ab + bc + ca &= r^2 + s^2 + 4Rr. \\ \text{Next,} \quad a^2 + b^2 + c^2 &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= 4s^2 - 2\left(r^2 + s^2 + 4Rr\right) \\ &= 2\left(s^2 - r^2 - 4Rr\right). \end{aligned}$$

Distance from vertex to incentre.

$$AI^2 = \frac{r^2}{\sin^2 A/2}.$$

This follows from the definition of sine of an angle (applied to the triangle whose vertices are A, I and the point of contact of the incircle with side AB). The same triangle yields:

$$\begin{aligned} AI^2 &= r^2 + (s-a)^2 = r^2 + s^2 - a(b+c) \\ &= r^2 + s^2 - (ab+bc+ca) + bc = bc - 4Rr, \end{aligned}$$

using the identity proved above for $ab + bc + ca$.

Formulas connecting the cosines of the angles.

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R},$$
$$\cos A \cos B + \cos B \cos C + \cos C \cos A = \frac{r^2 + s^2 + 4Rr}{4R^2} - \left(1 + \frac{r}{R}\right),$$
$$\cos A \cos B \cos C = \frac{s^2 - r^2 - 4R^2 - 4Rr}{4R^2}.$$

The first formula may be proved using geometrical arguments and the theorem of Ptolemy. We shall prove the second formula and the third formula using the first one. We use the fact that since $A + B + C = 180°$, we have $\cos A = -\cos(B + C)$, and similarly for $\cos B$ and $\cos C$. Hence:

$$-\left(\cos A + \cos B + \cos C\right) = \cos(B + C) + \cos(C + A) + \cos(A + B).$$

Now we expand the terms on the right side. We obtain:

$$\begin{aligned}&\cos(B+C)+\cos(C+A)+\cos(A+B)\\&=(\cos B\,\cos C-\sin B\,\sin C)+\text{two other such terms}\\&=(\cos B\,\cos C+\cos C\,\cos A+\cos A\,\cos B)-(\sin B\,\sin C+\sin C\,\sin A+\sin A\,\sin B)\,.\end{aligned}$$

The first bracketed term is the one we want. The second bracketed term is

$$\begin{aligned}\sin B\,\sin C+\sin C\,\sin A+\sin A\,\sin B&=\frac{bc}{4R^2}+\frac{ca}{4R^2}+\frac{ab}{4R^2}\\&=\frac{bc+ca+ab}{4R^2}\\&=\frac{r^2+s^2+4Rr}{4R^2}.\end{aligned}$$

It follows that

$$\cos A\,\cos B+\cos B\,\cos C+\cos C\,\cos A=\frac{r^2+s^2+4Rr}{4R^2}-\left(1+\frac{r}{R}\right).$$

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