HOW TO PROVE IT

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In the episode of "How To Prove It" that appeared in the March 2018 issue of *At Right Angles*, we studied a number of characterisations of a parallelogram; they were also listed in the article on parallelograms elsewhere in that issue. Three assertions had been made in the article without proof. We provide the proofs in this article.

The following question was posed in the article 'Parallelogram' in the March 2018 issue of this magazine: *What characterises a parallelogram?* In other words: *What minimal properties must a quadrilateral have for us to be sure that it is really a parallelogram?* (See [3] and [4].)

The basic definition of a parallelogram is: *A plane four-sided figure whose opposite pairs of sides are parallel to each other.* That is, a plane four-sided figure $ABCD$ is a parallelogram if and only if $AB \parallel CD$ and $AD \parallel BC$ (see Figure 1). An alternative definition, framed in the language of transformations, is: *A parallelogram is a quadrilateral with rotational symmetry of order* 2.

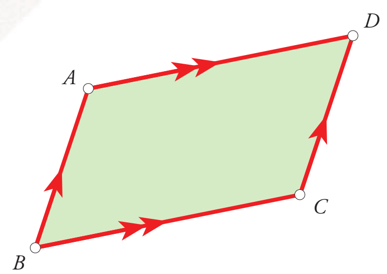


Figure 1.

In the articles mentioned above, we listed five properties possessed by a parallelogram and asked in each case whether the property in question characterises a parallelogram; i.e., *if a planar quadrilateral possesses that property, is it then necessarily a parallelogram*? We reproduce that list below, retaining the numbering from the original articles.

Keywords: Parallelogram, characterisation, congruence, one-way implication

Which of the following properties characterises a parallelogram?

(5) If $ABCD$ is a parallelogram, then each of its diagonals divides it into a pair of triangles with equal area. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that each of its diagonals divides it into two triangles that have equal area, then is ABCD necessarily a parallelogram?*

(6) If $ABCD$ is a parallelogram, then $AB = CD$ and $AD \parallel BC$. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that* $AB = CD$ *and* $AD \parallel BC$*, then is ABCD necessarily a parallelogram?*

(7) If $ABCD$ is a parallelogram, then $AB = CD$ and $\measuredangle A = \measuredangle C$. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that* $AB = CD$ *and* $\measuredangle A = \measuredangle C$*, then is ABCD necessarily a parallelogram?*

(8) If $ABCD$ is a parallelogram, then the sum of the squares of the sides equals the sum of the squares of the diagonals. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

then is ABCD necessarily a parallelogram?

(9) If $ABCD$ is a parallelogram, then the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point. Does this characterise a parallelogram? That is: *If ABCD is a planar quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, then is ABCD necessarily a parallelogram?*

We then showed—through counterexamples—that statements 6 and 7 *do not* provide characterisations of a parallelogram, and added (without proof) that statements 5, 8 and 9 do provide the asked-for characterisations. We now provide the proofs.

Proofs of statements 5, 8 and 9

Theorem 5. *If ABCD is a planar quadrilateral such that each of its diagonals divides it into two triangles with equal area, then ABCD is a parallelogram.*

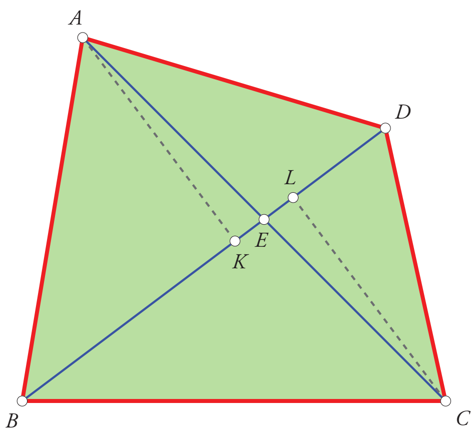


Figure 2.

Proof. Let's start with the statement that diagonal BD bisects the quadrilateral into two triangles with equal area. Figure 2 shows the relevant picture. Drop perpendiculars AK and CL from A and C to diagonal BD.

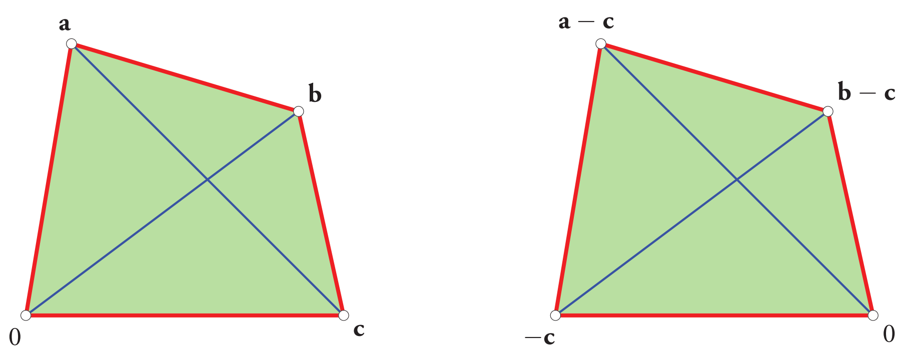
Since triangles ABD and CBD have equal area, it follows from the formula for area of a triangle ("half base times height") that $AK = CL$.

Now consider $\triangle AKE$ and $\triangle CLE$. It is easy to see that they are congruent to each other ('ASA congruence'). Hence $AE = CE$. In other words, *diagonal BD bisects diagonal AC*. This conclusion follows from the hypothesis that BD bisects the quadrilateral into two triangles with equal area.

In the same way, from the hypothesis that diagonal AC bisects the quadrilateral into two triangles with equal area, we deduce that diagonal AC bisects diagonal BD.

This means that the two diagonals bisect one another. It is well-known (and trivial to prove) that this property implies that $ABCD$ is a parallelogram. ■

Proof using vectors. Taking one vertex of the quadrilateral to be the origin, let the position vectors of the other three vertices be, in cyclic order, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (Figure 3a). The claim that these four points are the vertices of a parallelogram may then be replaced by the equivalent claim that $\mathbf{b} = \mathbf{a} + \mathbf{c}$. So this is what we must prove, under the hypothesis that each of the diagonals bisects the quadrilateral into two parts with equal area.



(a) (b)

Figure 3.

Let's start with the condition that the diagonal joining the points 0 and $\mathbf{b}$ divides the quadrilateral into two parts with equal area (Figure 3a). Using the vector formula for area, we arrive at the following condition:

$$\frac{1}{2}(\mathbf{b} \times \mathbf{a}) = \frac{1}{2}(\mathbf{c} \times \mathbf{b}).$$

Cancelling common factors in bringing all the terms to one side, we deduce that

$$\mathbf{b} \times (\mathbf{a} + \mathbf{c}) = 0,$$

which implies that $\mathbf{b}$ is parallel to $\mathbf{a} + \mathbf{c}$. Hence $\mathbf{b} = k(\mathbf{a} + \mathbf{c})$ for some real number k.

Now we consider the other condition, that the diagonal joining the points $\mathbf{c}$ and $\mathbf{a}$ too divides the quadrilateral into two parts with equal area. Shifting the origin so that it falls at vertex $\mathbf{c}$, the picture assumes the form shown in Figure 3b. Using the vector formula for area, we deduce that

$$\mathbf{a} - \mathbf{c} = m(\mathbf{b} - \mathbf{c} - \mathbf{c}),$$

for some real number m. (We do not have to go through the steps all over again; we can use the final equality deduced earlier.) Substituting for $\mathbf{b}$ from above, we obtain

$$\mathbf{a} - \mathbf{c} = m(k(\mathbf{a} + \mathbf{c}) - 2\mathbf{c}),$$

$$\therefore\ \mathbf{a} - \mathbf{c} = mk\mathbf{a} + m(k-2)\mathbf{c}.$$

Since **a** and **c** are linearly independent vectors, we may equate their coefficients on the two sides of the above equality and solve for m and k. We obtain $mk = 1$ and $mk - 2m = -1$, and these equations yield $m = 1$ and $k = 1$. Hence $\mathbf{b} = \mathbf{a} + \mathbf{c}$, and the required conclusion has been proved. ■

Remark. It is usually the case that a vector-based proof achieves the desired goal far more efficiently and compactly than an approach based on pure geometry. In the above example, however, the opposite seems to be the case!

Theorem 8. *If ABCD is a planar quadrilateral such that*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

then ABCD is a parallelogram.

Proof. We shall make use of a well-known theorem first proved in the second century AD: the theorem of Apollonius. (Indeed, we shall use the theorem as many as six times!)

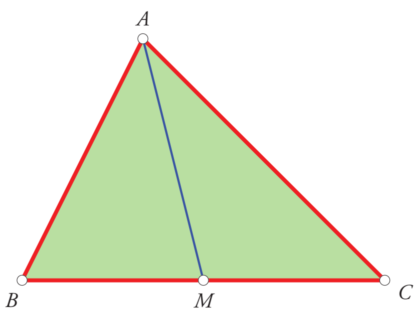


Figure 4.

The theorem provides a relation between the lengths of the medians and the lengths of the sides of a triangle: if ABC is any triangle and AM is a median (i.e., M is the midpoint of side BC; see Figure 4), then

$$AB^2 + AC^2 = 2\left(AM^2 + BM^2\right).$$

Otherwise put: if a, b, c are the lengths of the sides of the triangle, and m_a, m_b, m_c the lengths of the medians (named in the symmetric manner), then $4m_a^2 = 2b^2 + 2c^2 - a^2$ (with similar relations for m_b and m_c).

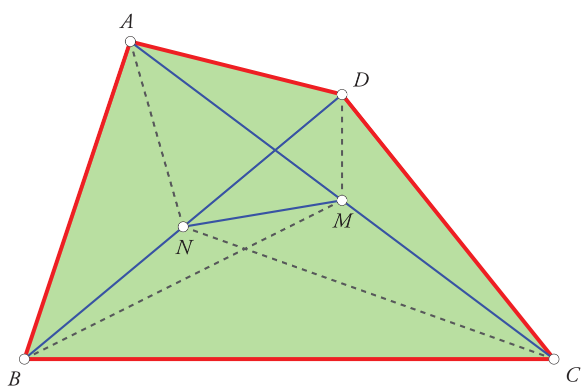


Figure 5.

With these preliminaries in place, we may set out the proof as follows. Let M and N be the midpoints of diagonals AC and BD respectively. Repeated application of the theorem of Apollonius (to triangles ABD, *DAC*, *CDB* and *BCA* respectively) yields the following equalities:

$$AB^2 + AD^2 = 2AN^2 + 2BN^2,$$
$$DA^2 + DC^2 = 2DM^2 + 2AM^2,$$
$$CD^2 + CB^2 = 2CN^2 + 2DN^2,$$
$$BC^2 + BA^2 = 2BM^2 + 2CM^2.$$

Adding up the corresponding sides of all these equalities, on the left side we obtain twice the sum of the squares of the four sides, i.e.,

$$2\left(AB^2 + BC^2 + CD^2 + DA^2\right).$$

On the right side, note that $BN^2 = DN^2$, and so $2BN^2 + 2DN^2 = 4BN^2 = BD^2$. Similarly, $2AM^2 + 2CM^2 = 4AM^2 = AC^2$. This takes care of four of the terms on the right side.

To process the sum of the remaining terms, $2\left(AN^2 + DM^2 + CN^2 + BM^2\right)$, we invoke the theorem of Apollonius yet again, applying it to triangles *NAC* and *MDB* respectively:

$$AN^2 + CN^2 = 2MN^2 + 2CM^2,$$
$$BM^2 + DM^2 = 2MN^2 + 2BN^2.$$

Hence:

$$AN^2 + DM^2 + CN^2 + BM^2 = 4MN^2 + 2CM^2 + 2BN^2$$
$$\therefore\ 2\left(AN^2 + DM^2 + CN^2 + BM^2\right) = 8MN^2 + AC^2 + BD^2.$$

It therefore follows that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2.$$

and *this equality will hold for any quadrilateral ABCD whatsoever*. (Note that this is an interesting result in its own right! It is far from obvious.)

It follows that if $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ for a given quadrilateral *ABCD*, then it must be that $MN = 0$; in other words, *M* and *N* are coincident, i.e., the midpoints of the diagonals are coincident. But this means that the diagonals bisect one another. As is well-known, this condition implies that *ABCD* is a parallelogram. We have reached the desired conclusion. ■

Proof using vectors. As earlier, we may attempt a vector-based proof; but this time, it does turn out to be more compact and also more insightful than the pure geometry proof!

Let the position vectors of the vertices of quadrilateral *ABCD* be **a**, **b**, **c**, **d** respectively. For any vector **v**, we shall (for convenience) use the symbol $\mathbf{v}^2$ to denote the product $\mathbf{v} \cdot \mathbf{v}$.

The sum of the squares of the lengths of the sides is equal to

$$(\mathbf{a}-\mathbf{b})^2 + (\mathbf{b}-\mathbf{c})^2 + (\mathbf{c}-\mathbf{d})^2 + (\mathbf{d}-\mathbf{a})^2$$
$$= 2\left(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2\right) - 2\left(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{a}\right).$$

Next, the sum of the squares of the lengths of the diagonals is equal to

$$(\mathbf{a}-\mathbf{c})^2 + (\mathbf{b}-\mathbf{d})^2$$
$$= \left(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2\right) - 2\left(\mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}\right).$$

Hence, the sum of the squares of the lengths of the sides minus the sum of the squares of the lengths of the diagonals is equal to

$$\left(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2\right) - 2\left(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{a}\right) + 2\left(\mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}\right).$$

A close and searching look at this expression reveals that it is a perfect square! Namely, it is equal to

$$[(\mathbf{a}+\mathbf{c}) - (\mathbf{b}+\mathbf{d})]^2.$$

This may be rewritten more revealingly as

$$4\left[\left(\frac{\mathbf{a}+\mathbf{c}}{2}\right) - \left(\frac{\mathbf{b}+\mathbf{d}}{2}\right)\right]^2.$$

This expression represents four times the square of the length of the segment that connects the midpoints of the two diagonals. This proves the identity proved earlier (using the theorem of Apollonius): $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$. The rest of the proof is the same as earlier. ■

Theorem 9. *If ABCD is a planar quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, then ABCD is a parallelogram.*

Vector-based proof. As it happens, in this particular case, we know only a vector-based proof – but it is a most elegant and pleasing proof!

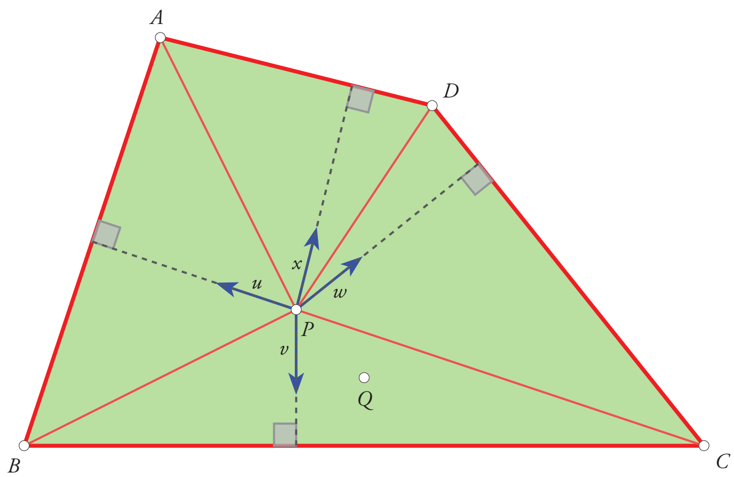


Figure 6.

Let $ABCD$ be a quadrilateral with the given property (see Figure 6), namely: the sum of the perpendicular distances from any point within the quadrilateral to the sides of the quadrilateral is the same. Let P be an arbitrary point within the quadrilateral. Join P to the vertices A, B, C, D and draw unit vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ from P perpendicular respectively to the sides AB, BC, CD, DA of the quadrilateral, as shown.

Now, observe that the perpendicular distances from P to AB, BC, CD, DA are respectively the dot products $\mathbf{PA}\cdot\mathbf{u}$, $\mathbf{PB}\cdot\mathbf{v}$, $\mathbf{PC}\cdot\mathbf{w}$, $\mathbf{PD}\cdot\mathbf{x}$. Therefore:

$$\mathbf{PA}\cdot\mathbf{u} + \mathbf{PB}\cdot\mathbf{v} + \mathbf{PC}\cdot\mathbf{w} + \mathbf{PD}\cdot\mathbf{x} = k,$$

where k is some constant. If Q is some other point within the quadrilateral and we repeat the construction described above (i.e., draw unit vectors from Q perpendicular to the sides and join Q to the vertices; note that *the unit vectors are the same as earlier*), then we have:

$$\mathbf{QA}\cdot\mathbf{u} + \mathbf{QB}\cdot\mathbf{v} + \mathbf{QC}\cdot\mathbf{w} + \mathbf{QD}\cdot\mathbf{x} = k.$$

Hence by subtraction we get:

$$\mathbf{PQ}\cdot\mathbf{u} + \mathbf{PQ}\cdot\mathbf{v} + \mathbf{PQ}\cdot\mathbf{w} + \mathbf{PQ}\cdot\mathbf{x} = 0,$$

i.e.,

$$\mathbf{PQ} \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x}) = 0.$$

Now imagine Q moving along a small circle centred at P (staying entirely within the quadrilateral at all times). The above equality is true for every such point Q. Since the dot product is 0 for vectors pointing in every possible direction, it must be that $\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x}$ is the zero vector, identically.

If the sum of four vectors is the zero vector, then we can lay them out edge-to-edge to get a closed shape (i.e., we return to the starting point in the end); in this case, we get a quadrilateral. Since the four vectors have equal length, the quadrilateral is in fact a rhombus. But the opposite sides of a rhombus are parallel to each other; hence $\mathbf{u}$ and $\mathbf{w}$ are collinear, as are $\mathbf{v}$ and $\mathbf{x}$. *But this implies that AB and CD are parallel to each other, as are BC and DA.* Hence *ABCD* is a parallelogram. ■

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