ClassRoom

Generalising the Divisibility Tests of 9 and 11

A RAMACHANDRAN

Divisibility tests for 9 and 11 are generally taken up at upper primary school level. The test for 9 seems to remain in people's consciousness long after they leave school and move on to other things. The assertion that the sum of the digits of any multiple of 9 is itself a multiple of 9 seems to give a mystical aura to the number 9. The test for 11 is more involved and does not seem to be retained as well, but other patterns in the multiples and powers of 11 are widely appreciated.

The explanations why these tests work calls for a higher level of understanding and does not seem to be part of the curriculum though it may appear under 'enrichment.' Both these tests rely on the fact that we follow the base 10 or decimal system. The place values in a decimal system are 1, 10, 10^2, 10^3,10^4,… Now all these numbers leave a remainder of 1 when divided by 9. So when we add the digits we are actually adding the remainders under division by 9.

For example, when $241 = 2 \times 100 + 4 \times 10 + 1$ is divided by 9, the remainder is $2 + 4 + 1 = 7$. To see why, observe that on division by 9, 200 leaves a remainder of 2, 40 leaves a remainder of 4, and 1 leaves a remainder of 1, so when 241 is divided by 9, the remainder is $2 + 4 + 1$.

In the case of division by 11, the situation is slightly different. 1, 10^2, 10^4, … leave remainders of 1 when divided by 11 ($1 = 0 \times 11 + 1$; $100 = 9 \times 11 + 1$; $10000 = 909 \times 11 + 1$ and so on). On the other hand, 10, 10^3,10^5, … are all one short of a multiple of 11 ($10 = 1 \times 11 - 1$; $1000 = 91 \times 11 - 1$; $100000 = 9091 \times 11 - 1$, etc.). The sum of the digits positioned at the even powers of 10 is thus the sum of the remainders (excesses) under division by 11, while the sum of the digits positioned at the odd powers of 10 is the sum of the shortfalls (deficiencies) from the next multiple of 11. Therefore, the number

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under test is divisible by 11 if and only if the difference of these sums is a multiple of 11. (Note that 0 too is a multiple of 11.)

For example, when $143 = 1 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$ is divided by 11, the remainder is $3 - 4 + 1 = 0$. To see why, observe that on division by 11, 1×10^2 and 3×10^0 leave remainders of +1 and +3, while 4×10^1 is 4 short of a multiple of 11. So 143 leaves remainder $1 + 3 - 4 = 0$ when divided by 11. Thus, it is divisible by 11. In the same way, 987 leaves remainder $7 - 8 + 9 = 8$ when divided by 11; it is not divisible by 11.

So these tests of divisibility hinge on the fact that 9 is one less than 10 and 11 is one more than 10, the base of the decimal system. Then a similar situation should arise in other bases too. Let us check it out with base 6 system. Here 5 and 7 play the role of 9 and 11, respectively. The place values in base 6 are 1, 6, 36, 216, 1296, 7776, etc. (That is, in base 10 representation; in base 6 representation they would just be 1, 10, 100, etc.) All these leave a remainder of 1 under division by 5. Even powers of 6, i.e.,1, 36, 1296,….leave a remainder of 1 under division by 7, while odd powers of 6, namely, 6, 216, 7776,….are all one short of a multiple of 7.

We now see if we can show the same for the general case, say base n. The place values here would be 1, n, n^2, n^3, n^4, n^5, … Now $n - 1$ and $n + 1$ take the roles of 9 and 11. Note the following:

- $1 = (n - 1) \times 0 + 1$;
- $n = (n - 1) \times 1 + 1$;
- $n^2 = (n - 1)(n + 1) + 1$; this follows from $n^2 - 1 = (n - 1)(n + 1)$;
- $n^3 = (n - 1)(n^2 + n + 1) + 1$; this follows from $n^3 - 1 = (n - 1)(n^2 + n + 1)$;
- $n^4 = (n - 1)(n^3 + n^2 + n + 1) + 1$; and so on.

In general, $n^k - 1$ is divisible by $n - 1$ for all numbers k.

So all powers of n leave a remainder of 1 under division by $n - 1$, analogous to the situation with 9 in base 10.

Further,

- $n = (n + 1) \times 1 - 1$;
- $n^3 = (n + 1)(n^2 - n + 1) - 1$; this follows from $n^3 + 1 = (n + 1)(n^2 - n + 1)$;
- $n^5 = (n + 1)(n^4 - n^3 + n^2 - n + 1) - 1$; and so on.

In general, $n^k + 1$ is divisible by $n + 1$ for all odd numbers k.

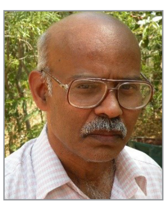
Moreover,

- $1 = (n + 1) \times 0 + 1$;
- $n^2 = (n + 1)(n - 1) + 1$;
- $n^4 = (n + 1)(n^2 + 1)(n - 1) + 1$;
- $n^6 = (n + 1)(n^2 - n + 1)(n^3 - 1) + 1$; and so on.

In general, $n^k - 1$ is divisible by $n + 1$ for all even numbers k.

Thus alternate place values are one more and one less than a multiple of $n + 1$, analogous to the situation with 11 in base 10.

It is hoped that this explanation will not take away from the sense of wonder or magic that a young student may associate with these numbers.



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.