At Right Angles

A Resource for School Mathematics

RATIO

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INTRODUCTION

How does one introduce a topic like ratio, which is so widely present in daily life and so intimately connected with human experiences? Our cherished cultural achievements are permeated with it: music is full of ratios, as is art. Our daily existence involves cooking and shopping, and these are filled through and through with the usage of ratio. Shadows, which are present with us all through the day, offer a visual depiction of ratios in action. In mathematics as a subject, the notion of ratio is embedded into many topics – sometimes in an obvious way, at other times not so. Fractions, scale drawing, enlargement, trigonometry, tables, linear equations… are all illustrations of this.

Considering that ratio is linked in an essential way to so many concepts in mathematics, it is important to lay a strong foundation and develop the necessary conceptual base when teaching this topic. A basic error that often arises while studying ratio is the application of *additive thinking* to a context that requires *multiplicative thinking*. This is an error that students make quite frequently while solving ratio problems. There is a clear need to point this out in various ways and by using different examples, so that students clearly see that ratio and proportion are based on a multiplicative relationship.

Often the problems that students encounter in textbooks are limited and become repetitive.

In addition, lack of exposure to sufficient variety inhibits students from noticing the proportional situation in varied contexts. They may not develop the ability to distinguish between proportional and non-proportional situations. Again, most problems involve whole number scale factors and students get flustered when they come across real life problems that do not fall into that category; i.e. the scale factor is a fraction. Examples taken up should cover diverse areas and bring in fractional scale factors as well.

While students encounter the topic in different forms as they go to higher grades, we need to reiterate for ourselves what we expect the student to achieve during the introductory phase.

Expectations

- Understand the relationship of ratios to fractions: While fractions compare part to the whole, ratios compare both part-to-part and part-to-whole.
- Understand the importance of order in reading and writing ratios.
- Explore and generate equivalent ratios.
- Understand that a proportion is based on a multiplicative relationship and generalise the numerical relationship, i.e., if a : b $: : c : d$, then $ad = bc$.
- Recognise proportional and non-proportional contexts.
- Understand the usage of unitary method in relation to proportion problems.
- See rate as the ratio of two quantities having different units (rate being a special kind of ratio).
- Divide a number into a given ratio.
- Solve a proportion problem using different strategies and diagrams.

So, where do we begin?

Students have an intuitive understanding of Ratio and Proportion. It may be good to begin there, to let them use their intuitive understanding and then help them to notice the principles they have applied, study the principles, experiment with them and articulate them. I shall lay out gradually what I mean by 'studying and experimenting'.

Activities 1 to 5 and Game 1 build on this intuitive understanding.

Keywords: Ratio, proportion, fraction, part to part, part to whole, equivalent ratio, unitary method

ACTIVITY 1

Objective: To draw out intuitive multiplicative thinking through a visual poser.
Materials: Hexagon pieces

Lay out a design pattern for the students as shown in Figure 1.

If I have 12 yellow hexagons, how many blue hexagons do I need to make designs as in Figure 1?

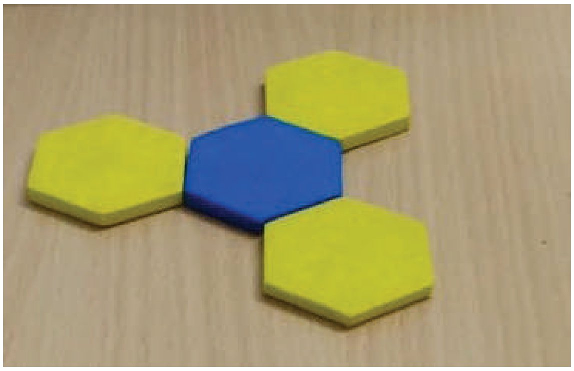


Figure 1

If I have 6 blue hexagons, how many brown hexagons do I need to make designs as in Figure 2?

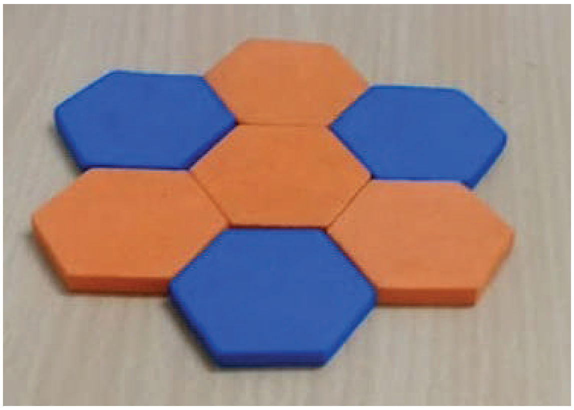


Figure 2

ACTIVITY 2

Objective: To point out multiplicative thinking used by students.

Pose a problem based on the classroom situation. Let students use their intuition in responding to it.

Tanvi and Mouli are solving problems. For every two problems that Tanvi solves, Mouli solves three.

If Tanvi solved eight problems, how many did Mouli solve?

Let the students think it over for a few minutes and allow them to come up with their responses. Ask them to explain their answers.

Some may give the right answer, 12, but it is possible that some student will respond by saying 9.

'Mouli solved nine (six more than three) as Tanvi solved eight (six more than two)'.

How do the other students respond to such reasoning? Do they see the flaw in the reasoning?

Once the discussion is over, the teacher should reiterate the significance of the words 'for every' and then point out to the students the multiplicative reasoning that they applied.

For every two problems that Tanvi solved, Mouli solved three. If Tanvi solved eight problems which is four times 2, then Mouli will solve four times 3, which is 12.

ACTIVITY 3

Objective: To develop intuitive understanding of proportion through enlargement.
Materials: Square dot paper

Give a drawing of a letter of the alphabet or a numeral, as shown in the figure.

Ask students to double it. Were they able to do it correctly?

Did their drawing seem right to them?

What kind of errors did they make?

What could have caused the errors?

Ask them to triple it. Did they find any difficulty?

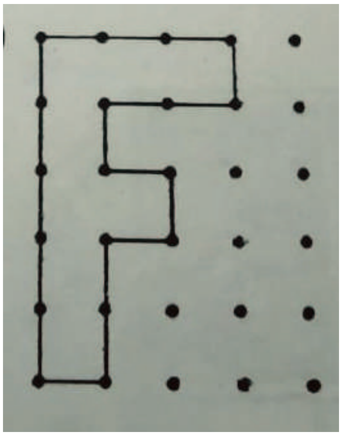


Figure 3

ACTIVITY 4

Objective: To develop intuitive understanding of proportion through model building.
Materials: Unifix Cubes

Give the class a 2 × 2 × 2 cube.

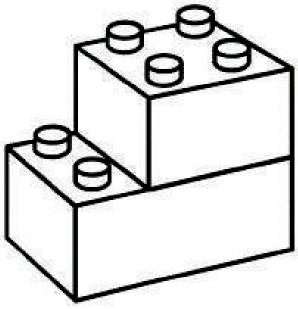
Ask them to double it.

How did they approach the problem?

How many cubes will they need? Did they forget to double any dimension?

Ask the class to triple this shape.

Did they find this challenging?



ACTIVITY 5

Objective: To spot similar shapes

Materials: Sets of rectangles, isosceles or right-angled triangles and trapeziums (each set to contain one pair of similar figures). Note that we do not permit use of tools of measurement.

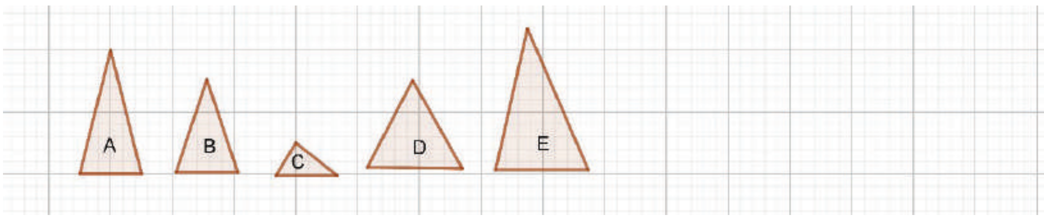


Figure 6

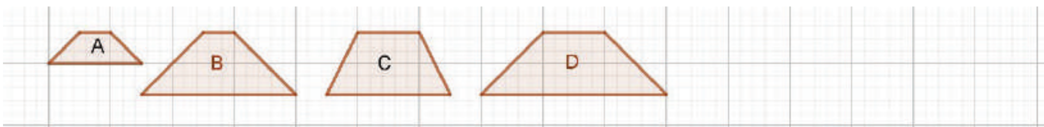


Figure 7

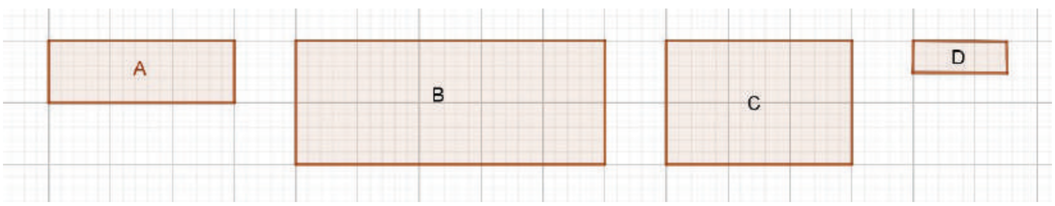


Figure 8

The teacher can prepare these using square grid papers and mount them on plain card sheet. Each shape can be labelled using alphabets. We must take care in selecting the sizes, so that spotting similarity is neither too easy nor too difficult.

Divide the students into three groups. Give the sets one at a time to the groups so that they can discuss and select the pair that looks similar to them. They can use either the plain side of the shapes or the side with square grid in deciding about similarity. By rotation, each group looks at the three sets and notes down the similar pairs in each set.

At the end of the activity, all groups share their findings and present their reasoning. Now verify the answer using a ruler.

You could keep the challenge level high by insisting that they use only the plain side of the shapes.

Follow this up by a worksheet that requires testing for similarity.

Are the figures in each set similar?

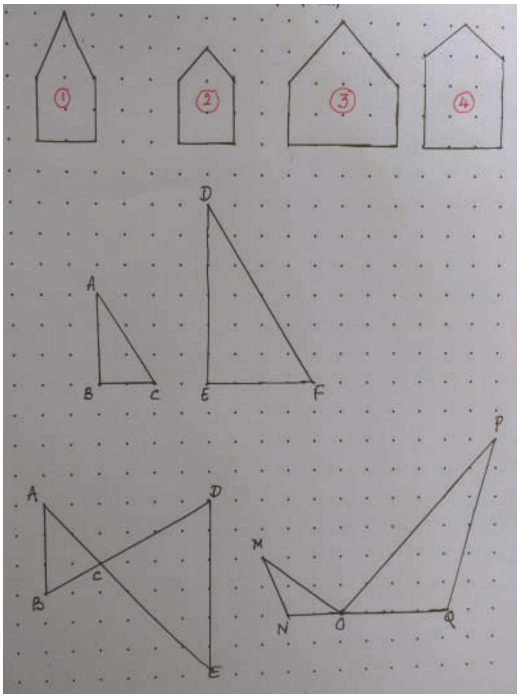


Figure 9

GAME 1

Objective: To spot equivalence of ratios and group them

Materials: 16 Cards with appropriate pictures to demonstrate equivalent ratios (for example, four equivalent ratios of 3:1, 1:4, 3:4 and 2:3)

Here are two pictures for equivalent ratios of 1:2

Figure 10

Figure 10.1

Figure 10.2

Divide the class into two teams. Ask each team, one after the other, to sort the cards and group them. If they manage to group the four sets correctly, then they get four points.

Analyse how they have grouped them. Ask the students the basis of their groupings.

Does everyone agree with the reasoning behind the grouping?

ACTIVITY 6

Objective: To introduce what a ratio is and to show how we read and write ratios.

Materials: Classroom

Point to the furniture arrangement in the classroom.

Figure 11

In this picture, each table has two stools with it.

The ratio of tables to stools is 1:2.

Teach students how the ordering of words affects the way we write a ratio.

For instance, in this picture the ratio of stools to tables is 2:1.

Get students to describe this cube design using ratios.

What is the ratio of black to grey cubes?

How do the black cubes compare with the white cubes?

Are the students able to use multiplicative thinking?

They can record their findings in sentences.

There are twice as many _____ cubes as _____ cubes.

As a second step, they can write it in the form of ratio statements.

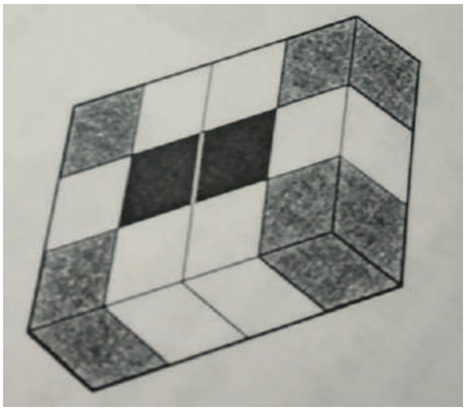


Figure 12

ACTIVITY 7

Objective: To study and experiment with a given ratio.
Materials: Cubes or counters

Here are 2 yellow cubes and 4 black cubes.

The ratio of yellow to black is 2:4.

There are 2 times as many black cubes as yellow cubes.

If you add 2 cubes of each colour, will the black cubes still be 2 times the yellow cubes? What is the new ratio of yellow to black?

What will happen if you take away 1 cube of each colour from the original pattern? What is the new ratio of yellow to black?

What will happen if you double each colour? What is the new ratio of yellow to black?

What will happen if you halve each colour? What is the new ratio of yellow to black?

Do the students form any conclusions based on this experimentation?

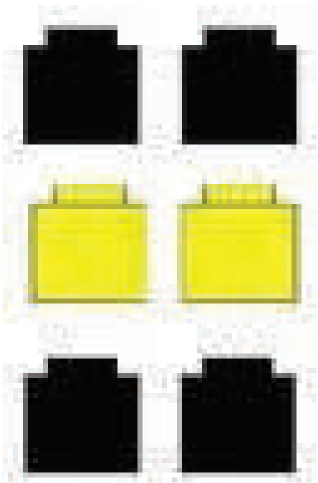


Figure 13

ACTIVITY 8

Objective: Explore the multiplicative relationship of ratios by generating equivalent ratios.
Materials: Square grid paper

The teacher needs to show the students the relationship of fractions to ratios. She also needs to point out the differences. They may not have recognised before that every fraction is in fact a ratio.

While studying fractions, we compare the part to the whole, but while studying ratios we compare both part-to-part as well as part-to-whole.

Also in the case of fractions, the order in which we write the numbers is fixed, the numerator is the part and the denominator is the whole. In the case of ratios, there is no such fixed order.

The teacher can show the close connection between equivalent fractions and equivalent ratios through drawings.

This figure shows a part-to-whole ratio.

Students should record the work as shown in this figure to internalise the fraction, ratio connection.

Students also need to understand that, just as with fractions, we can reduce ratios by common factors of the quantities, to make the simplest form.

Example: 4:12 can be divided by 4 on both sides to give 1:3.

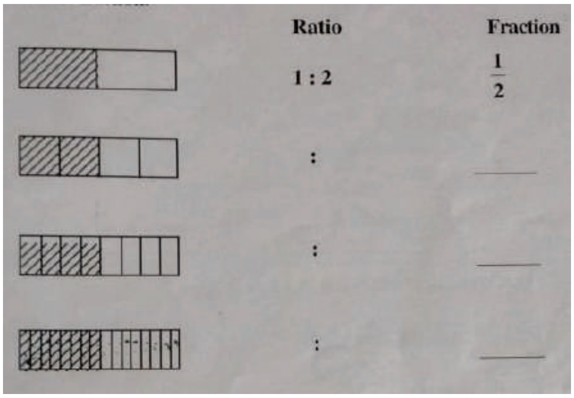


Figure 14

8:10 can be divided by 2 on both sides to give 4:5.

Discuss part-to-whole and part-to-part ratios in the same context to make the difference clear.

A class has 18 boys and 12 girls.

The ratio of boys to girls is 18:12 or 3:2. This is a part-to-part ratio.

The ratio of boys to total number of students is 18:30 or 3:5. This is a part to whole ratio.

ACTIVITY 9

Objective: To show that a proportion is based on a multiplicative relationship and generalise the numerical relationship.
Materials: Five equivalent ratios of 3:1, 1:4, 3:4 and 2:3 written out on separate cards

Distribute the cards to students. Ask them to sort the ratio cards and group them. Through inspection, they will spot the equivalent ratios and group them. Verify that they are right.

The teacher can now draw their attention to the multiplicative relationship within each group of ratios, explain what proportion means and point out that the ratios in each group are in proportion.

Raise the question: "If we take any two ratios which are in proportion, how do the numbers relate to each other? Say, 6:8 to 15:20." One obvious relationship is that we can reduce both pairs to a common ratio. Is there any other relationship? Can we see a relationship if we write them as fractions? Help the students notice that 6 times 20 equals 8 times 15. Ask them whether this happens with every pair of ratios in proportion.

Through discussion, the teacher can generalise the procedure, i.e., when a:b::c:d, then ad=bc.

The class can then discuss practical examples involving proportion like recipes, heartbeat, physical exercise like runs.

However, at the same time, the teacher must also expose the students to non-proportional contexts so that they are able to identify and differentiate between proportional and non-proportional contexts.

A simple example of a non-proportional context is the comparison of the ages of two people.

Aarav is 12 years old. His sister Ami is 6 years old. How do these numbers compare? Aarav is twice as old as Ami. What will be the case 6 years later? Would Aarav be twice as old as Ami?

One day I'll catch up! My Dad is four times as old as I am. When will he be three times as old? When will he be twice as old? When will I catch up?

An interesting exploration to do would be to examine how the scale factor behaves, as the daughter and father get older. Can the students see why this happens?

Reflection: What line of inquiry can the student now pursue? What tasks can the teacher set?

ACTIVITY 10

Objective: Understand the usage of unitary method in relation to proportion problems.
Materials: Price lists of products, food items etc.

Unitary method, i.e., calculating per unit is a very common way of solving proportion problems. Many students find it natural and easy-to-understand and take to it without any difficulty.

Example. 3 chocolate bars cost Rs. 45. How much will 7 bars cost?

They may reason it out as, if 3 bars cost Rs. 45, then 1 costs Rs. 15, so 7 bars will cost Rs. 105.

It seems reasonable and efficient to solve it this way.

Is the unitary method always the best way?

If 500 balloons cost Rs. 2745, what will be the cost of 800 balloons?

Finding the cost of one chocolate bar from the price of three bars seems perfectly sensible, but finding the cost of one balloon from the cost of 500 balloons does not. A far more workable method here would be to find the cost of 100 balloons and then to multiply by 8.

Naturally, we need to modify the unitary method according to the situation.

Figure 15

ACTIVITY 11

Objective: Aids to solving problems **(modelling)**
Materials: Square paper

In the initial stages, the teacher can give problems accompanied by visuals.

In the second stage, the student can model a given problem.

Modelling is an important tool for solving word problems.

Ex. In a class, the ratio of girls to boys is 2:3. If the number of students in the class is 30 what is the number of boys? What is the number of girls?

Here is one way of modelling it.

For every 2 girls there are 3 boys in the class.

Each such group contains 5 students.

How many groups of 5 in 30?

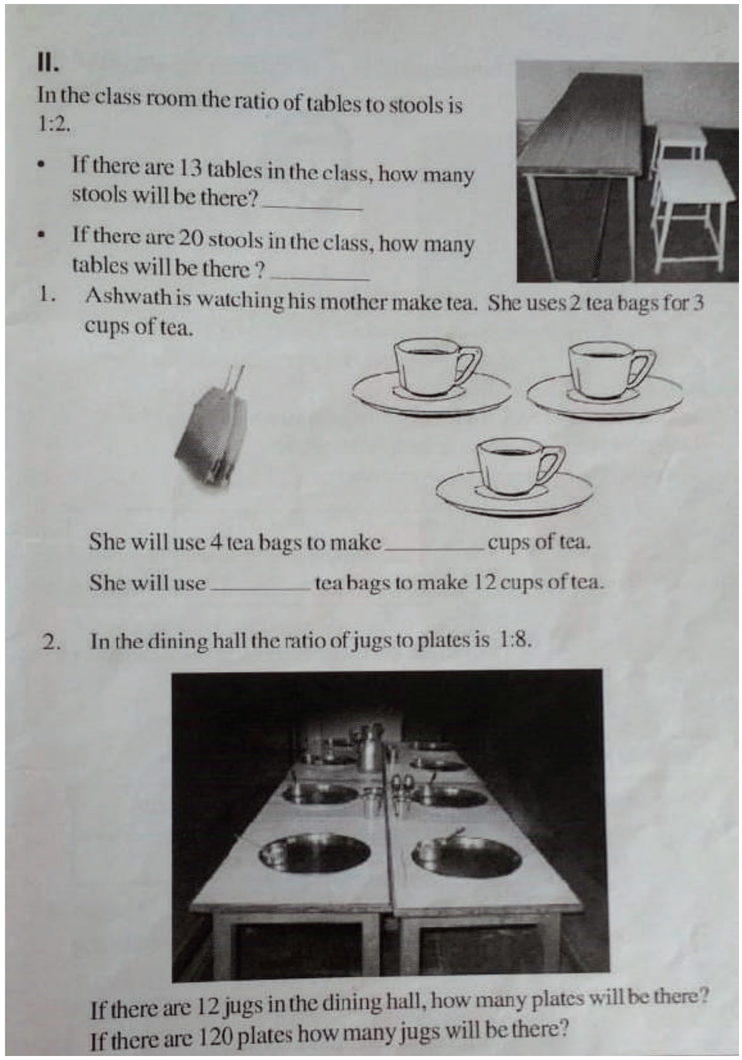
6 such groups.

Therefore, in 6 such groups, the number of girls is 12 and the number of boys is 18.

Reflection: An important question while solving problems is this. *What imagery does the student use to understand a problem?*

Can we teach imagery? Is not imagery a subjective experience that may be difficult to articulate?

At the same time, helping students to find their own imagery requires that the teacher expose them to visuals, multiple strategies and approaches to problem solving. Imagery is often closely linked to the way a concept has been developed. Otherwise, simplifying ratios or finding proportions can become highly mechanical.



II.

In the class room the ratio of tables to stools is 1:2.

- If there are 13 tables in the class, how many stools will be there? ________
- If there are 20 stools in the class, how many tables will be there ? ________

1. Ashwath is watching his mother make tea. She uses 2 tea bags for 3 cups of tea.

She will use 4 tea bags to make ________ cups of tea.

She will use ________ tea bags to make 12 cups of tea.

2. In the dining hall the ratio of jugs to plates is 1:8.

If there are 12 jugs in the dining hall, how many plates will be there?
If there are 120 plates how many jugs will be there?

Figure 16

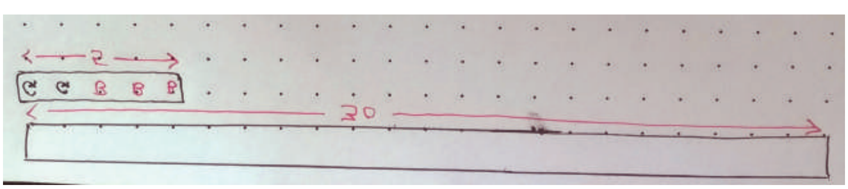


Figure 17

ACTIVITY 12

Objective: Aids to solving problems **(rephrasing)**

Another important aid in solving ratio problems is the ability to rephrase a problem in ratio language. What is 'ratio language?' It involves the use of phrases and words such as 'for each', 'for every', 'per', 'each time' and so on.

In the beginning, it is good practice for students to rewrite the information of a problem in ratio language.

Help the students to rephrase the problems.

Example: At an ice-cream shop, the ratio of chocolate cones sold to vanilla cones sold was 4:3. If the shopkeeper sold 84 ice-cream cones in a day, how many chocolate cones did he sell?

For every 4 chocolate cones sold, there were 3 vanilla cones sold.

This makes a group of 7 objects.

How many groups of 7 are there in 84? 12 such groups.

Therefore, the number of chocolate cones will be 12 times 4, i.e., 48.

Writing Mathematics

Students should be encouraged to write out their reasoning for a solution.

Example: One-fourth of the class is absent today. 30 students are present today. How many students are in that class?

"If one-fourth of the class is absent, then three-fourth of the class is present. Three-fourth stands for 30. Therefore, one-fourth stands for 10. So the total number of students is 40.

ACTIVITY 13

Objective: Aids to solving problems **(table making)**

In a library, the number of non-fiction books is one-fourth the number of fiction books.

How many fiction books would be there if there are 50 non-fiction books?

How many non-fiction books would be there if there are 120 fiction books?

Non-fiction	20		50		100
fiction	80	240		120	

Figure 18

What approaches would you use for these problems?

Aunty Elizabeth decides to give her nephew and niece Rs.1500, to be shared between them. Because the niece is the older of the two, she decides that she must divide the money between the niece and the nephew in the ratio 5:3. How much does each child get?

For every 4 kilometres that Jaleel jogged, Kian jogged 3 kilometres. If Jaleel jogged 1 km, how far would Kian have jogged?

A student finished 8 homework problems in class. If the ratio of problems finished to the problems still left was 4: 1, how many homework problems did she have in total?

Occasionally, pose an **open-ended** question.

Think!

The ratio of two even numbers is 3:7. What are they?

It takes 3 minutes to boil 1 egg. How many minutes does it take to boil 2 eggs?

GAME 2

Objective: Interpret given ratios and construct a feasible design.
Materials: Square grids of 5×5 size

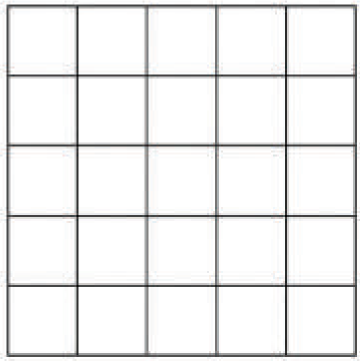


Figure 19

25 tiles of 4 different colours

The teacher constructs a design and shares it with the students using ratios.

Examples.

The ratio of green to red is 5:4.

The ratio of green to blue is 10:1.

The ratio of red to yellow is 4:3.

The pattern has two lines of symmetry.

Are the students able to construct a design that satisfies all these conditions?

After playing the game a few times, students can form teams and create challenges for one another.

Creating a pattern that lends itself to good clues is an educative and challenging experience in itself, particularly if the solution is unique.

Extension: We can also use grids of size 4 × 4.

ACTIVITY 14

Objective: Studying ratio applications (maps from an atlas)
Materials: Atlas or map charts

Teacher can help students read a map and observe the way scale is used in the map, the way the scale is denoted on the map and the need for accuracy in the representation.

Students can apply their knowledge to estimate the distances between pairs of places and the lengths of rivers. They can also compare the lengths of the coastlines of different states of India. And so on.

ACTIVITY 15

Objective: Meeting real life ratio challenges (scale drawings)
Materials: Measuring tapes and scales

Each time a teacher draws on the blackboard or writes on the blackboard, she is implicitly using a ratio. A 20 cm long pencil when drawn on the board may become 30 cm long.

That is an example of ***enlargement*** or multiplication, by a scale factor of 1½.

Maps and scale drawings, in contrast, depict reduction in scale.

Divide the students into groups and ask each group to make a scale drawing of a classroom, library or the playground.

Discuss what would be a sensible scale to use.

In the process of making a scale drawing, they will encounter fractional ratios and at times need to do rounding off.

ACTIVITY 16

Objective: Highlight the need for a common unit or conversion in a ratio.

Example. Madhu says, "I will take 10 days to complete my project." Vishal says, "I need 2 weeks."
Ashok described the ratio of time needed by Madhu and Vishal as 10:2.

What do you think?

Let the students discuss the implication of a 10:2 ratio in this context.

If it is wrong, why is it wrong? What is the right way of expressing it?

ACTIVITY 17

Objective: See rate as the ratio of two quantities having different units
(rate is a special kind of ratio).
Materials: Price list, purchase bills, electricity bills, travel time

What is a rate? A rate compares different kinds of measures, such as rupees per kilogram (cost per unit bought), kilometres per hour (speed of a vehicle), rupees per day (wages for a worker), heartbeats per minute (state of health). So many aspects of our life involve some kind of rate.

Plotting of rate in a table as shown demonstrates the multiplicative relationship very clearly.

We can also show this data as a graph.

Time (h)	1	2	3	4	5
Distance (mi)	60	120	180	240	300

Figure 20

ACTIVITY 18

Objective: Meeting real life ratio challenges (recipes)
Materials: Actual recipes!

No discussion on ratios can end without discussing the use of ratios in recipes, on which our very survival rests!

Strong coffee? Weak coffee?

Teacher can distribute four recipes for making coffee. The students have to arrange them according to the strength of the decoction.

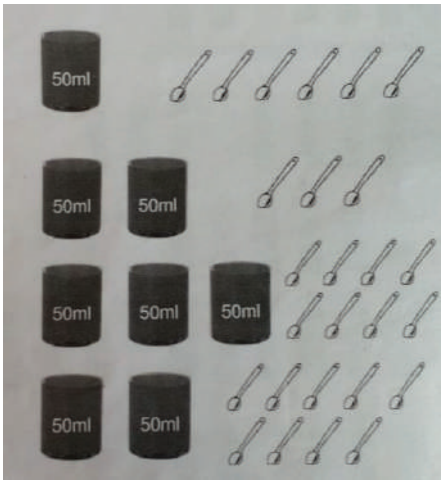


Figure 21

If coffee is not welcome, we can substitute it with orange juice!

Ordering four recipes requires comparison of four different strengths.

However, we may also make comparisons within a recipe.

Here is one such example.

Comparison within one recipe between the quantity of juice and water.

To make a jug of orange juice, I use two glasses of orange juice and five glasses of water.

What happens to the strength of the mixture if I add an extra glass of water?

What happens if I add an extra glass of orange juice?

What happens if I add one of each, two of each, etc.?

ACTIVITY 19

Objective: Solving proportion problems as algebraic equations.
Materials: Word problems

In a bag of red and green sweets, the ratio of red sweets to green sweets is 3:4. If the bag contains 120 green sweets, how many red sweets are there?

For every 3 red sweets there are 4 green sweets.

Number of red sweets = y

Number of green sweets = 120

Hence, $3:4::y:120$

$3 \times 120 = 4 \times y$

$4y = 360$

$y = 90$

Project Ideas

- Study of shadows
- Sports data
- Human body (comparative study)
- Currency exchange rates
- Microscope enlargements

History: The ancient Greeks originally posed this problem:

'How can a line be divided into two parts so that the ratio of the larger part to the smaller is the same as the ratio of the whole line to the larger part?'

Historically, what kind of ratio problems did people study? What challenges would they have faced which led to the development of topics like trigonometry? It would be of interest to the students and teachers to study this question.

To close, a few questions...

- How does one support and scaffold the students' learning?
- Has one exposed the students to various strategies so that we have been able to achieve the intended goals or learning outcomes?
- By studying incorrect solutions, can one understand the misconceptions that the students hold and so address them?

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