A Triangle Area Problem

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Problem

We study the following area problem. In Figure 1 we see a $\triangle ABC$ in which segments have been drawn from two of its vertices (B, C) to the opposite sides. These segments divide the triangle into four parts — three triangles and a quadrilateral. The areas of the four parts are denoted by a, b, c, d, as shown in the figure. The problem is: *Find a relationship between a, b, c, d*. (Note that no other information is given; in particular, the area of the triangle is not given.) Use this relationship to find c in terms of a, b, d.

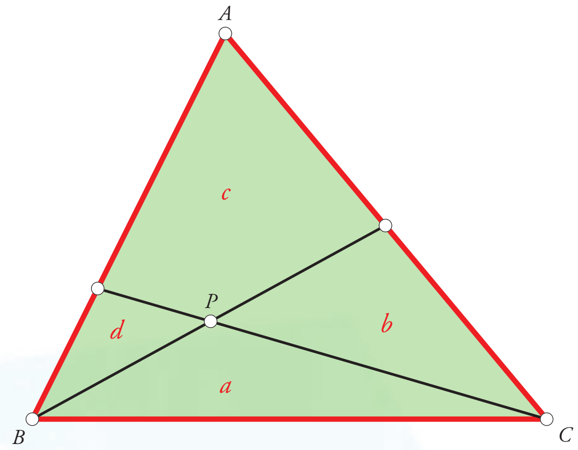

Figure 1.

We urge the reader to tackle this challenging problem before reading further.

Solution

Areal ratio principle. We make repeated use of a simple idea (we refer to it as the 'areal ratio principle'). Consider any triangle ABC (Figure 2). Let D be any point on BC and let P be any point on AD. Join PB, PC.

Keywords: Triangle, quadrilateral, area, ratio, componendo-dividendo

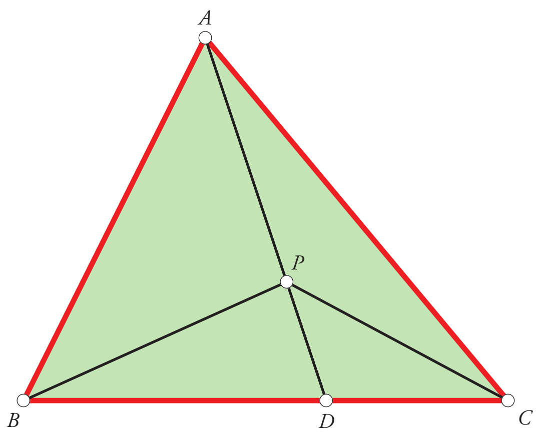

Figure 2.

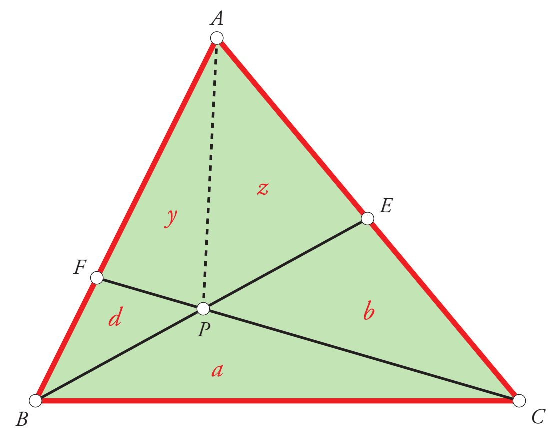

Figure 3. Here $c = y + z$

Then we have the following equalities:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} = \frac{BD}{CD} = \frac{\text{Area of } \triangle PBD}{\text{Area of } \triangle PCD},$$

and therefore,

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} = \frac{\text{Area of } \triangle PBD}{\text{Area of } \triangle PCD} = \frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ACP}.$$

Here we are using a simple arithmetical principle: if we have two equal fractions $\frac{a}{b} = \frac{c}{d}$ (where $a \neq c$, $b \neq d$), then we also have the equalities

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d} = \frac{a+c}{b+d}.$$

More generally, we have: if $\frac{a}{b} = \frac{c}{d}$ (where $a \neq c$, $b \neq d$), then

$$\frac{a}{b} = \frac{c}{d} = \frac{a+kc}{b+kd} \quad \text{for any real number } k \neq -\tfrac{b}{d}.$$

The proposition is easy to prove; we leave the details to the reader.

Applying the areal ratio principle to the given problem

In order to apply this principle to our problem, we draw an additional line segment (Figure 3) and denote by y, z the areas of the two triangles into which the quadrilateral (with area c) is divided.

Invoking the areal-ratio principle stated above, we obtain the following relations:

$$\frac{y}{d} = \frac{z+b}{a}, \qquad \frac{y+d}{a} = \frac{z}{b}.$$

Hence:

$$ay - dz = bd,$$
$$-by + az = bd.$$

These equations are readily solved for the unknowns y, z; we obtain, after a couple of steps,

$$y = \frac{bd(a+d)}{a^2 - bd}, \qquad z = \frac{bd(a+b)}{a^2 - bd}.$$

Since $c = y + z$, we get

$$c = \frac{bd(2a+b+d)}{a^2 - bd}.$$

It follows that the desired relation between a, b, c, d is the following:

$$\boxed{c\left(a^2 - bd\right) = bd(2a+b+d).}$$

This is the required solution.

Remarks

Looking at the above relation closely, we may make some interesting remarks.

- Since the quantities a, b, c, d are necessarily positive, it follows from the above relation that

$$a^2 - bd > 0, \quad \text{i.e.,} \quad a^2 > bd.$$

It is not immediately obvious why this inequality should be true, other than through this indirect route!

- The above relation expresses c uniquely in terms of a, b, d. However, what if we are given b, c, d and wish to find a? It would appear now that we have a quadratic equation at hand (in the unknown quantity a):

$$a^2c - 2abd - bd(b + c + d) = 0.$$

This raises the intriguing possibility of obtaining two different values for a. If this truly transpired, it would indeed be very curious, geometrically. However, the possibility cannot happen; for, the sum and product of the two values of a arising from the above quadratic equation are equal to

$$\frac{2bd}{c}, \quad -\frac{bd(b+c+d)}{c},$$

respectively. Of these, the first quantity is positive, whereas the second quantity is clearly negative. Hence one of the values of a must be positive, while the other value must be negative. The negative value is not admissible in a real situation.

Much the same thing happens if we are given a, c, d and wish to find b. The resulting equation is again quadratic:

$$b^2d + bd(2a + c + d) - a^2c = 0.$$

It is easy to check that the sum of the two values of b arising from this equation is positive, whereas the product is negative. Hence one of the values of b must be positive, while the other value must be negative.

By symmetry, we expect that this behaviour will be replicated if we are given a, b, c and wish to find d. The reader is invited to check out this statement.

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