Area covered by Two Intersecting Circles

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Websites and focus interest groups are a good source for interesting problems. But it's rarely that one gets down to solving these; more often they go into a to-do list. We hope that the solution presented here will encourage you to try more of these.

Look at the steps of the process: Visualisation, definition of the problem, connection to known formulas and then good old mathematical processing. Problem solved!

Here is a challenging problem in mensuration. It was posed in the LinkedIn group "Math, Math Education, Math Culture" by one of the members, Robert Lewis.

Figure 1 shows portions of two circular disks: ω, with centre $A\,(0,0)$ and radius 4, and Γ, with centre $B\,(8,8)$ and radius 8. Their boundaries intersect at points C, D.

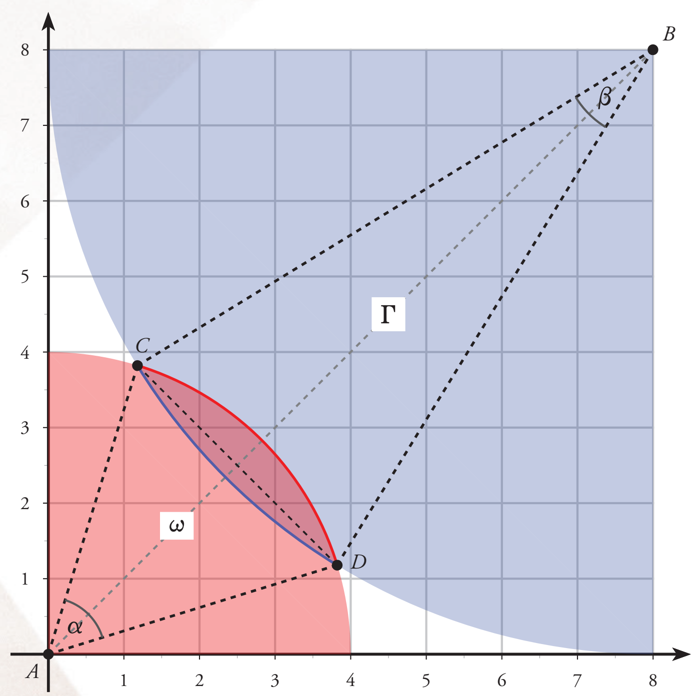

Figure 1.

Keywords: Circle, disk, lune, kite, area, cosine rule

The problem is to find the area of the region enclosed within both the disks, i.e., the area of the lune enclosed by the two circular arcs CD. (A 'lune' is any region bounded by two circular arcs, or by a line and a circular arc.)

Let $\alpha = \measuredangle CAD$ and $\beta = \measuredangle CBD$. The area of a sector with central angle θ in a circle of radius r is $\frac{1}{2}r^2\theta$ (here θ is measured in radians), and the area of an isosceles triangle with equal sides r enclosing an angle θ is $\frac{1}{2}r^2 \sin\theta$. Hence the required area is

$$\frac{4^2}{2}(\alpha - \sin\alpha) + \frac{8^2}{2}(\beta - \sin\beta). \tag{1}$$

It remains to find the angles α and β and then make the necessary substitutions. Consider $\triangle BAD$. Since $AD = 4$, $BD = 8$ and $AB = 8\sqrt{2}$, the cosine rule gives:

$$\cos\frac{\alpha}{2} = \frac{4^2 + (8\sqrt{2})^2 - 8^2}{2\cdot 4\cdot 8\sqrt{2}} = \frac{5\sqrt{2}}{8},$$

$$\cos\frac{\beta}{2} = \frac{8^2 + (8\sqrt{2})^2 - 4^2}{2\cdot 8\cdot 8\sqrt{2}} = \frac{11\sqrt{2}}{16}.$$

Hence:

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 2\left(\frac{5\sqrt{2}}{8}\right)^2 - 1 = \frac{9}{16},$$

$$\cos\beta = 2\cos^2\frac{\beta}{2} - 1 = 2\left(\frac{11\sqrt{2}}{16}\right)^2 - 1 = \frac{57}{64},$$

and:

$$\sin\alpha = \sqrt{1 - \frac{9^2}{16^2}} = \sqrt{\frac{16^2 - 9^2}{16^2}} = \sqrt{\frac{7\times 25}{16^2}} = \frac{5\sqrt{7}}{16},$$

$$\sin\beta = \sqrt{1 - \frac{57^2}{64^2}} = \sqrt{\frac{64^2 - 57^2}{64^2}} = \sqrt{\frac{7\times 121}{64^2}} = \frac{11\sqrt{7}}{64}.$$

So the area of the lune is:

$$\frac{4^2}{2}\left(\cos^{-1}\frac{9}{16} - \frac{5\sqrt{7}}{16}\right) + \frac{8^2}{2}\left(\cos^{-1}\frac{57}{64} - \frac{11\sqrt{7}}{64}\right). \tag{2}$$

To get the numerical value we need a calculator (or 'tables'). We find the area to be $1.173 + 0.555 \approx 1.728$ square units. Note that $\cos^{-1} 9/16$ and $\cos^{-1} 57/64$ must be evaluated in radians.

A variation

The proposer (Robert Lewis) next modified the problem to make it more challenging, as follows. Leaving disk ω unchanged, he replaces Γ by a disk with radius r and centre (r, r). For the two disks to intersect and the lune to have positive area, r must lie between $4(\sqrt{2}-1)$ and $4(\sqrt{2}+1)$, as can be shown by solving an appropriate pair of equations. This is a nice problem in itself, but we leave it to you to solve.

Here is his challenge:

Find some values of r for which the area of the lune is a nice quantity.

As 'nice' is not a well defined mathematical attribute, we must give it a suitable interpretation. We shall take it to mean: "expressible in terms of familiar quantities". ('Familiar' is not a mathematical attribute either! — but we mean: frequently met numbers like the integers, the rationals, numbers like $\sqrt{2}$, $\sqrt{3}$, π, e, $\ln 2$, $\ln 3$, and so on. Let us keep the notion loose, deliberately, and not attempt to make it more precise than this.) With this interpretation, we plunge in and look for suitable values of r.

We use the same notation and the same figure, except that now $B = (r, r)$ and the radius of Γ is r. Let $\alpha = \measuredangle CAD$ and $\beta = \measuredangle CBD$. The area of the lune is, as earlier:

$$\frac{4^2}{2}(\alpha - \sin\alpha) + \frac{r^2}{2}(\beta - \sin\beta). \tag{3}$$

We must find α and β. Consider $\triangle BAD$. Since $AD = 4$, $BD = r$ and $AB = r\sqrt{2}$, the cosine rule gives:

$$\begin{aligned} \cos\frac{\alpha}{2} &= \frac{4^2 + (r\sqrt{2})^2 - r^2}{2\cdot 4\cdot r\sqrt{2}} = \frac{(r^2+16)\sqrt{2}}{16r}, \\ \cos\frac{\beta}{2} &= \frac{r^2 + (r\sqrt{2})^2 - 4^2}{2\cdot r\cdot r\sqrt{2}} = \frac{(3r^2-16)\sqrt{2}}{4r^2}. \end{aligned} \tag{4}$$

From these we get expressions for $\cos\alpha$, $\cos\beta$, $\sin\alpha$, $\sin\beta$. Substituting these in turn into (3) and doing a substantial amount of simplification (but we will spare you the details), we arrive at the following expression $f(r)$ for the area of the lune:

$$\begin{aligned} &\frac{4^2}{2}\cos^{-1}\frac{r^4-32r^2+256}{64r^2} + \frac{r^2}{2}\cos^{-1}\frac{5r^4-96r^2+256}{4r^4} \\ &\quad - \frac{\sqrt{-(r^4-96r^2+256)}}{2}. \end{aligned} \tag{5}$$

For what values of r will $f(r)$ assume a form expressible in terms of known quantities? Surely, those for which the $\cos^{-1}$ term simplifies to a recognizable form. This means that we want the expressions

$$P = \frac{r^4-32r^2+256}{64r^2}, \qquad Q = \frac{5r^4-96r^2+256}{4r^4}$$

to take values like 0, $1/2$, $1/\sqrt{2}$, $\sqrt{3}/2$, $(\sqrt{5}-1)/4$ and 1, for which the inverse cosine has a recognizable form. (This list can be extended, but we have not attempted to do so.)

So now we equate each of P and Q to 0, $1/2$, $1/\sqrt{2}$, $\sqrt{3}/2$, $(\sqrt{5}-1)/4$ and 1, solve for r, and examine the outcome. At the end of this exercise (for which we omit the details; it is rather tedious!), we find only two suitable candidate values of r, both of which are obtained by solving the equation $P = 1/2$:

$$r = 2\sqrt{6} - 2\sqrt{2}, \qquad r = 2\sqrt{6} + 2\sqrt{2}.$$

Substituting these values into the expression for $f(r)$ and simplifying the expressions, here is what we get:

- If $r = 2\sqrt{6} - 2\sqrt{2}$, the area of the lune is
$$\pi\left(16 - \frac{20}{\sqrt{3}}\right) - 8.$$
- If $r = 2\sqrt{6} + 2\sqrt{3}$, the area of the lune is
$$\frac{4\left(\pi(4+\sqrt{3}) - 6\sqrt{3} - 6\right)}{3}.$$

Inasmuch as 'nice' is a term which can be interpreted in different ways, the question posed at the start is open-ended and will doubtless permit many more such answers.

Closing remark

What exactly is 'nice'? As already noted, this notion is not well defined. And yet, we do have a vague sense of its meaning. When we evaluate an integral (say), we are happier with 2 as an answer than with 2.1; we are happier with π than with 3.140; we are happier with $\sqrt{2}$ than with $\sqrt{2.1}$. Probably you will agree with the author that the above answers are nicer than the answer we got for the original problem, $\frac{4^2}{2}\left(\cos^{-1}\frac{9}{16} - \frac{5\sqrt{7}}{16}\right) + \frac{8^2}{2}\left(\cos^{-1}\frac{57}{64} - \frac{11\sqrt{7}}{64}\right)$. So, implicitly, we do seem to operate with a sense of what is nice and what is not so nice! But let us leave it at that. The point was to have a bit of fun and nothing more!

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