Problem Corner

A Pythagoras-style Diophantine Equation and its Solution

$\mathcal{C} \otimes \mathcal{M} \alpha \mathcal{C}$

The Pythagorean equation $x^2 + y^2 = z^2$ (to be solved over the positive integers $\mathbb{N}$) is a much-studied one; many articles have appeared in this magazine alone, devoted to this equation. A close relative to this is the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ (which can be written as $x^{-1} + y^{-1} = z^{-1}$; in this form, its similarity to the Pythagorean equation is readily seen), and this too has been studied many times in *At Right Angles*.

In this note, we study another equation which visually resembles the Pythagorean equation and which too is required to be solved over the positive integers:

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{z}}. \tag{1}$$

Write

$$x = m^2 a, \quad y = n^2 b, \quad z = k^2 c, \tag{2}$$

where m, n, k are positive integers and a, b, c are 'square-free' positive integers, i.e., they are not divisible by any square number greater than 1. (So a, b, c are products of *distinct* prime numbers.) Any positive integer can be uniquely written in this form, i.e., as a product of a perfect square and a square-free positive integer. Making these substitutions, we get:

$$\frac{1}{m\sqrt{a}} + \frac{1}{n\sqrt{b}} = \frac{1}{k\sqrt{c}},$$

$$\therefore \quad km\sqrt{ac} + kn\sqrt{bc} = mn\sqrt{ab}. \tag{3}$$

Squaring both sides of (3), we get:

$$k^2 m^2 ac + 2mnck^2\sqrt{ab} + k^2 n^2 bc = m^2 n^2 ab.$$

Keywords: Diophantine equation, rational number, integer, prime number, square-free number

From this relation, we deduce that $2mnck^2\sqrt{ab}$ is an integer, and therefore that $\sqrt{ab}$ is a rational number. But if the square root of an integer is a rational number, then it is an integer. Hence $\sqrt{ab}$ is an integer. We know that in the prime factorisations of a and b, each prime occurs just once. If we combine this condition with the deduction that $\sqrt{ab}$ is an integer, we realize right away that $a = b$.

Again, (3) can be written as

$$mn\sqrt{ab} - km\sqrt{ac} = kn\sqrt{bc}. \qquad (4)$$

Squaring both sides of (4), we get:

$$m^2n^2ab - 2knam^2\sqrt{bc} + k^2m^2ac = k^2n^2bc.$$

From this relation, we deduce (just as we did earlier) that $2knam^2\sqrt{bc}$ is an integer, therefore that $\sqrt{bc}$ is a rational number, therefore that $\sqrt{bc}$ is an integer, therefore that $b = c$. Hence $a = b = c$. A striking conclusion!

This means that $x = m^2a$, $y = n^2a$ and $z = k^2a$ for some positive integers m, n, k, a. Equation (1) now yields:

$$\frac{1}{m\sqrt{a}} + \frac{1}{n\sqrt{a}} = \frac{1}{k\sqrt{a}}, \qquad \therefore \ \frac{1}{m} + \frac{1}{n} = \frac{1}{k}. \quad (5)$$

It is remarkable that in attempting to solve the equation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{z}}$ over $\mathbb{N}$, we have ended up (essentially) with the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$, also to be solved over $\mathbb{N}$! We know very well how to solve this equation; all we need to do now is to invoke what we had discovered earlier.

A corollary to what we discovered above is the following: *If coprime positive integers x, y, z satisfy the relation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{z}}$, then each of x, y, z is a perfect square.* This is so because the condition that x, y, z are coprime forces $a = 1$, implying that $x = m^2$, $y = n^2$ and $z = k^2$. A neat result!

A specific example. Let us illustrate this by taking, say $z = 20$. So we seek all solutions (x, y) in positive integers to the equation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}$.

Since $20 = 2^2 \times 5$, what we showed earlier implies that $\sqrt{x}$ and $\sqrt{y}$ are integer multiples of $\sqrt{5}$. Let $x = 5m^2$ and $y = 5n^2$ where m and n are positive integers. Then the equation reduces to $\frac{1}{m} + \frac{1}{n} = \frac{1}{\sqrt{4}}$, i.e., $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$. We know very well how to solve this kind of equation! We have:

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2},$$

$$\therefore \ 2(m+n) = mn,$$

$$\therefore \ mn - 2(m+n) = 0,$$

$$\therefore \ (m-2)(n-2) = 4.$$

Since 4 can be written as a product of two positive integers in the following ways,

$$4 = 1 \times 4 = 2 \times 2 = 4 \times 1,$$

it follows that

$$(m-2, n-2) \in \{(1,4),\ (2,2),\ (4,1)\},$$

and hence that

$$(m, n) \in \{(3,6),\ (4,4),\ (6,3)\}.$$

Since $x = 5m^2$ and $y = 5n^2$, it follows that

$$(x, y) \in \{(45,180),\ (80,80),\ (180,45)\}.$$

Hence the equation $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}$ has these positive integer solutions and no more.

Much the same approach can be followed for any given value of z.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.