feature

Set Theory Revisited

As easy as PIE

The Principle of Inclusion and Exclusion – Part 1

Recall the old story of two frogs from Osaka and Kyoto which meet during their travels. They want to share a pie. An opportunistic cat offers to help and divides the pie into two pieces. On finding one piece to be larger, she breaks off a bit from the larger one and gobbles it up. Now, she finds that the other piece is slightly larger; so, she proceeds to break off a bit from that piece and gobbles that up, only to find that the first piece is now bigger. And so on; you can guess the rest. The frogs are left flat!

B Sury

We are going to discuss a simple but basic guiding principle which goes under the name *principle of inclusion and exclusion*, or PIE for short. Was it inspired by the above tale? Who knows . . . . The principle is very useful indeed, because counting precisely, contrary to intuition, can be very challenging!

An old formula recalled

Here is a formula which you surely would have seen many times: If A and B are two finite, overlapping sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|. \qquad (1)$$

Here, of course, the vertical bars indicate *cardinality*: $|A|$ is the cardinality of (or number of elements in) A, and so on. The formula is rather obvious but may be justified by appealing to the Venn diagram (see Figure 1).

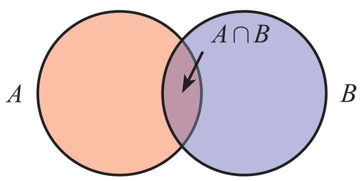

Figure 1

Once one has the basic idea, it is easy to generalize the formula to *three* overlapping finite sets A, B, C. In order to find the cardinality of $A \cup B \cup C$ we start naturally enough with an addition: $|A| + |B| + |C|$. But now several items have been counted twice, and some have even been counted thrice (those that lie in all three sets). So we compensate by subtracting the quantities $|A \cap B|$, $|B \cap C|$ and $|C \cap A|$. But now we have bitten off too much: the items originally in $A \cap B \cap C$ have been left out entirely (see Figure 2). So we compensate by putting these items back in, and now we have the correct formula:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|. \quad (2)$$

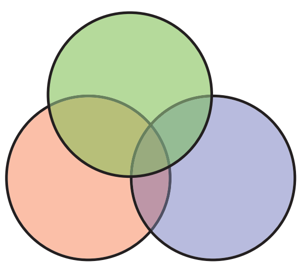

Figure 2

Generalizing the formula

How shall we generalize these formulas? We do so by considering the following problem. Suppose there are N students in a class and a fixed, finite number of subjects which they all study. Denote by N_1 the number of students who like subject #1, by N_2 the number of students who like subject #2, and so on. Likewise, denote by $N_{1,2}$ the number of students who simultaneously like the subjects 1 and 2, by $N_{2,3}$ the number of students who simultaneously like subjects 2 and 3, and so on. Similarly, denote by $N_{1,2,3}$ the number of students who simultaneously like subjects 1, 2, 3; and so on. Now we ask: Can we express, in terms of these symbols, the number of students who do not like *any* of the subjects? (There may well be a few students in this category!) We shall show that this number is given by

$$N - (N_1 + N_2 + \cdots) + \left(N_{1,2} + N_{2,3} + \cdots\right) - \left(N_{1,2,3} + \cdots\right) + \cdots . \quad (3)$$

Note the minus-plus-minus pattern of signs: we alternately subtract to avoid over counting, then add to compensate as we have taken away too much, then again subtract, and so on. The formula follows from a reasoning known as the *principle of inclusion and exclusion*, commonly abbreviated to 'PIE'.

Here is how we justify the formula. We start, naturally, by subtracting $N_1 + N_2 + \cdots$ from N. Now study the expression $N - (N_1 + N_2 + \cdots)$. The subtraction of $N_1 + N_2$ means that we have *twice* subtracted the number of students who like the 1st and 2nd subjects. To compensate for this, we must add $N_{1,2}$. Similarly we must add $N_{1,3}$, $N_{2,3}$, and so on.

However, when we add $N_{1,2} + N_{2,3} + N_{1,3} + \cdots$, we have included those who like the first three subjects (numbering $N_{1,2,3}$) *twice*. So we must subtract $N_{1,2,3}$. Similarly for other such terms. Proceeding this way, we get the right number by alternately adding and subtracting.

Divide and conquer counting

The PIE allows us to solve the following problem in which N is any positive integer. *Among the numbers* 1, 2, 3, . . . , N*, how many are not divisible by either* 2 *or by* 3*?*

Here's how we solve this problem. Among the given numbers the number of multiples of 2 is $[N/2]$. Here the square brackets indicate the *greatest integer function*, also called the *floor function*. The meaning is this: if x is a real number, then $[x]$ is the largest integer not greater than x. For example: $[5] = 5$, $[2.3] = 2$, $[10.7] = 10$, $[\sqrt{10}] = 3$, $[-2.3] = -3$, and so on. (Note the way the definition applies to negative numbers.)

Similarly, the number of multiples of 3 in the set $\{1, 2, 3, \ldots, N\}$ is $[N/3]$. So we subtract both these quantities from N. But the numbers divisible by both 2 and 3 (i.e., the numbers divisible by 6) have been subtracted twice, so we add back the number of multiples of 6, which is $[N/6]$. Hence the answer to the question is:

$$N - \left[\frac{N}{2}\right] - \left[\frac{N}{3}\right] + \left[\frac{N}{6}\right].$$

We solve the following in the same way: *Let N be any positive integer. Among the numbers*

1, 2, 3, . . . , N, *how many are not divisible by any of the numbers* 2, 3, 5*?*

By alternately "biting away" too much, then compensating, we see that the answer is

$$N - \left[\frac{N}{2}\right] - \left[\frac{N}{3}\right] - \left[\frac{N}{5}\right] + \left[\frac{N}{6}\right] + \left[\frac{N}{10}\right] + \left[\frac{N}{15}\right] - \left[\frac{N}{30}\right].$$

Here 30 is the LCM of 2, 3, 5 (if a number is divisible by 2, 3 and 5 then it must be divisible by 30; and conversely).

The general formula. From this reasoning we arrive at the following general formula. *If N is a positive integer, and $n_1, n_2, \ldots$ are finitely many positive integers, every two of which are relatively prime, then the number of elements of* $\{1, 2, 3, \ldots, N\}$ *which are not divisible by any of the numbers $n_1, n_2, \ldots$ is*

$$N - \left(\left[\frac{N}{n_1}\right] + \left[\frac{N}{n_2}\right] + \cdots\right) + \left(\left[\frac{N}{n_1 n_2}\right] + \left[\frac{N}{n_1 n_3}\right] + \left[\frac{N}{n_2 n_3}\right] + \cdots\right) - \cdots. \qquad (4)$$

You should now be able to provide the formal justification for the formula on your own.

Euler's totient function

There is a special case of the above formula which is of great interest in number theory. We consider the following problem.

For a given positive integer N, what is the number of positive integers not exceeding N which are relatively prime to N?

The numbers which are relatively prime to N are exactly those which are not divisible by any of the prime divisors of N. Let us denote the primes dividing N by $p, q, r, \ldots$. Now we apply the idea described in the last section. We conclude that the required number is:

$$N - \left(\frac{N}{p} + \frac{N}{q} + \frac{N}{r} + \cdots\right) + \left(\frac{N}{pq} + \frac{N}{qr} + \frac{N}{pr} + \cdots\right) - \cdots. \qquad (5)$$

By factoring out N we find that the resulting expression can be factorized in a convenient manner; we get the following:

$$N\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right)\cdots. \qquad (6)$$

For example, take $N = 30$. Since $30 = 2 \times 3 \times 5$, we see that the number of positive integers not exceeding 30 and relatively prime to 30 is

$$30\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 30 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 8.$$

This is easily checked. (The positive integers less than 30 and relatively prime to 30 are 1, 7, 11, 13, 17, 19, 23 and 29.)

Formula (6) defines the famous *totient function* which we associate with the name of Euler. The symbol reserved for this function is $\varphi(N)$. So we may write:

$$\varphi(N) = N \prod_{p|N}\left(1 - \frac{1}{p}\right), \qquad (7)$$

the product being taken over all the primes p that divide N; that is why we have written '$p \mid N$' below the product symbol. (The symbol $\prod$ is used for products in the same way that $\sum$ is used for sums.)

Corollary: a multiplicative property

The formula for $\varphi(n)$ gives us another property as a bonus — the property that Euler's totient function is *multiplicative*: if m and n are relatively prime positive integers, then $\varphi(mn) = \varphi(m)\varphi(n)$.

Example: Take $m = 4, n = 5, mn = 20$. We have: $\varphi(4) = 2, \varphi(5) = 4$; next, by applying formula (6) we get: $\varphi(20) = 20 \times 1/2 \times 4/5 = 8$. Hence we have $\varphi(20) = \varphi(4) \cdot \varphi(5)$.

It is an interesting exercise to prove this multiplicative property without using formula (6). (It *can* be done, by looking closely at the definition of the function.)

In closing: relation between GCD and LCM

To demonstrate how unexpectedly useful the PIE formula can be, we mention here a nice application of the formula. However we shall leave it as a question without stating the actual result, and discuss the problem in detail in a sequel to this article.

Here is the context. We all know the pleasing formula that relates the GCD ("greatest common divisor", also known as "highest common factor") and the LCM ("lowest common multiple") of any two positive integers a and b:

$$\text{GCD}\,(a, b) \times \text{LCM}\,(a, b) = ab. \qquad (8)$$

You may have wondered: The above formula relates the GCD and LCM of *two* integers a, b. What would be the corresponding formula for *three* integers a, b, c? For *four* integers a, b, c, d? . . .

In Part II of this article we use the PIE to find a generalization of formula (8). Alongside we discuss a problem about a seemingly absent-minded but actually mischievous secretary who loves mixing up job offers sent to applicants so that every person gets a wrong job offer (for which he had not even applied!), and another problem concerning placement of rooks on a chessboard. And, venturing into deeper waters, we also mention a famous currently unsolved problem concerning prime numbers.

Exercises

(1) Show how the factorization in formula (6) follows from formula (5).
(2) Explain how formula (7) implies that the totient function $\varphi(N)$ is multiplicative.
(3) Let N be an odd positive integer. Prove directly, using the definition of the totient function (i.e., with invoking the property of multiplicativity), that $\varphi(2N) = \varphi(N)$.
(4) What can you say about the family of positive integers N for which $\varphi(N) = N/2$? For which $\varphi(N) = N/3$?
(5) Try to find a relation connecting LCM (a, b, c) and GCD (a, b, c).

Further reading

- V Balakrishnan, *Combinatorics: Including Concepts Of Graph Theory* (Schaum Series)
- Miklos Bona, *Introduction to Enumerative Combinatorics* (McGraw-Hill)

B. SURY got his Ph.D. in Mathematics from TIFR (Mumbai). He has been with the Indian Statistical Institute (Bangalore) since 1999. He has been interested in expository writing at school and college level and in interacting with mathematically talented students. He is the regional co-ordinator for the Math Olympiad in Karnataka and a member of the editorial committee of the newsletter of the Ramanujan Mathematical Society. His research interests are in algebra and number theory. Mathematical limericks are an abiding interest. He may be contacted at sury@isibang.ac.in. His professional webpage is www.isibang.ac.in/~sury.

The quadratic was solved with ease.
The cubic and biquadratic did tease
but were solved not long ere.
It was the quintic which made it clear
that algebra developed by degrees !

– B. Sury