problem corner

Two Combinatorial Problems

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'Good' subsets

The following problem was posed in a recent mathematics contest:

> ***Problem.*** *Consider all three-element subsets of the set S given by*
>
> $$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$
>
> *Call a subset 'good' if the sum of its elements is a multiple of* 3*. Thus,* $\{2, 3, 7\}$ *is good, but not* $\{2, 3, 8\}$*. Find the number of good three-element subsets of S.*

We shall find the answer in two ways — a brute-force way, using complete enumeration, and then by a more subtle approach, using ideas from number theory.

It is always a good idea to have an idea what the answer will be, roughly, when we are computing any quantity. Here, the total number of three-element subsets of S is $\binom{10}{3}$ which equals

$$\frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120.$$

For each subset, if we add the elements of that subset and divide by 3, the remainder must be 0, 1 or 2. We wish to count the number of times the remainder 0 occurs. It seems reasonable to suppose that there will not be a big difference between the number of occurrences of each of the remainders, 0, 1 and 2. Accordingly we expect the answer to the problem to be close to one-third of 120, i.e., close to 40. Let us see if this is so.

Brute-force way. Let $\{a, b, c\}$ denote a potential good subset of S, with $a < b < c$. Then a can be any of the numbers 0, 1, 2, ..., 7. Let a take each of these values in turn. We shall count the number of pairs $\{b, c\}$ with $a < b < c$ such that $\{a, b, c\}$ is good, and then find the total number from these individual numbers.

- Suppose $a = 0$. Then $b + c$ must be a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{1, 2\}$, $\{1, 5\}$, $\{1, 8\}$, $\{2, 4\}$, $\{2, 7\}$, $\{3, 6\}$, $\{3, 9\}$, $\{4, 5\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6, 9\}$, $\{7, 8\}$. Hence there are **12** possibilities.

- Suppose $a = 1$. Then $b + c$ must be 1 less than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{2, 3\}$, $\{2, 6\}$, $\{2, 9\}$, $\{3, 5\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$, $\{5, 9\}$, $\{6, 8\}$, $\{8, 9\}$. Hence there are **10** possibilities.

- Suppose $a = 2$. Then $b + c$ must be 1 more than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{3, 4\}$, $\{3, 7\}$, $\{4, 6\}$, $\{4, 9\}$, $\{5, 8\}$, $\{6, 7\}$, $\{7, 9\}$. Hence there are **7** possibilities.

- Suppose $a = 3$. Then $b + c$ must be a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{4, 5\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6, 9\}$, $\{7, 8\}$. Hence there are **5** possibilities.
- Suppose $a = 4$. Then $b + c$ must be 1 less than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{5, 6\}$, $\{5, 9\}$, $\{6, 8\}$, $\{8, 9\}$. Hence there are **4** possibilities.
- Suppose $a = 5$. Then $b + c$ must be 1 more than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{6, 7\}$, $\{7, 9\}$. Hence there are **2** possibilities.
- Suppose $a = 6$. Then $\{b, c\}$ must be $\{7, 8\}$. Hence there is just **1** possibility.
- Suppose $a = 7$. Then $\{b, c\}$ must be $\{8, 9\}$. Hence there is just **1** possibility.

So the required number is $12 + 10 + 7 + 5 + 4 + 2 + 1 + 1 = \mathbf{42}$. Note that the answer is close to 40, as anticipated.

A subtler approach. Let's see if we can do better. We categorize the numbers from 0 to 9 according to the remainder left when they are divided by 3. We get three sets:

- Remainder 0: $A_0 = \{0, 3, 6, 9\}$
- Remainder 1: $A_1 = \{1, 4, 7\}$
- Remainder 2: $A_2 = \{2, 5, 8\}$

Now there are two ways in which the sum of three numbers can be a multiple of 3 (a bit of thinking will convince you why this must be true):

(i) The three numbers leave the same remainder under division by 3; for example, a sum like $2 + 5 + 8$ or $0 + 6 + 9$. The underlying reason is, of course, that the sum $1 + 1 + 1$ is a multiple of 3 (and hence also $2 + 2 + 2$; the sum $0 + 0 + 0$ is clearly a multiple of 3).

(ii) The three numbers leave three different remainders under division by 3; for example, a sum like $1 + 2 + 6$ or $2 + 6 + 7$. The underlying reason is, of course, that the sum $0 + 1 + 2$ is a multiple of 3.

It follows that there are just two kinds of three-element subsets which are good: (i) those for which the three elements are all from A_0, all from A_1, or all from A_2; (ii) those which have one element each from A_0, A_1 and A_2. Let us count these separately.

Since $|A_0| = 4$, $|A_1| = 3$ and $|A_2| = 3$, the number of three-element subsets of the first kind is the sum of the number of three-element subsets of A_0, the number of three-element subsets of A_1 and the number of three-element subsets of A_2; that is, the sum of $\binom{4}{3}$, $\binom{3}{3}$ and $\binom{3}{3}$. This equals $4 + 1 + 1 = 6$.

The number of three-element subsets of the second kind is even simpler to compute: it is equal to the product $4 \times 3 \times 3 = 36$. Note the use of the 'multiplication principle of counting' here.

Hence the total number of good three-element subsets is $6 + 36 = \mathbf{42}$. We have got the same answer as earlier.

There are yet other ways of solving this problem, but we leave them for you to find.

Greatest odd divisor

Each positive integer has a *greatest odd divisor*. For example:

- $10 = 5 \times 2$, so the greatest odd divisor of 10 is 5.
- $48 = 3 \times 2^4$, so the greatest odd divisor of 48 is 3.

It should be clear that each positive integer n can be written in just one way as the product of an odd integer and a power of 2, and that odd integer is the largest odd divisor of n.

Here is a curious problem posed in a recent Regional Math Olympiad, pertaining to the greatest odd divisor function:

> ***Problem.*** *Consider the following set S of n numbers:*
>
> $$S = \{n + 1, n + 2, n + 3, \ldots, 2n - 1, 2n\}.$$
>
> *For each number in this set, find its greatest odd divisor. Show that the sum of these numbers is n^2.*

The result looks quite astonishing, doesn't it? However a closer look reveals that it is nothing but an old friend in a very clever disguise: namely, the statement that the sum of the first n odd numbers equals n^2. For example, consider the case when $n = 6$. We have:

$$7 = 7 \times 2^0, \quad 8 = 1 \times 2^3, \quad 9 = 9 \times 2^0,$$
$$10 = 5 \times 2^1, \quad 11 = 11 \times 2^0, \quad 12 = 3 \times 2^2.$$

So the odd numbers which we have to sum are $7, 1, 9, 5, 11, 3$. Note that these are simply the numbers $1, 3, 5, 7, 9, 11$ in a permuted order, and their sum is 5^2.

But how do we show this in general? It is easier than it looks. Consider the numbers in $S = \{n+1, n+2, n+3, \dots, 2n-1, 2n\}$. For $i = 1, 2, \dots, n-1, n$, let $n+i$ be written as

$$n + i = a_i \times 2^{b_i},$$

that is, $a_i \times$ some power of 2, where a_i is some odd positive integer and b_i a non-negative integer. Obviously, we must have $a_i \leq n + i$. Also, $n + i \leq 2n$; therefore $a_i \leq 2n$. Since a_i is odd, it follows that a_i is one of the numbers $1, 3, 5, \dots, 2n-1$.

Next we ask: if it is possible for $a_i = a_j$ for a pair of unequal indices i and j? Suppose this is the case; say $a_i = a_j$ for some pair i, j (here $i \neq j$). Let a denote the common value of a_i, a_j. Then we have, by supposition:

$$n + i = a \times 2^{b_i},$$
$$n + j = a \times 2^{b_j}.$$

It cannot be that $b_i = b_j$, for this would mean that $n + i = n + j$, i.e., $i = j$; but we had supposed that $i \neq j$. Hence $b_i \neq b_j$. This means that one of the b's is larger than the other one.

Suppose that $b_j > b_i$. Then b_j exceeds b_i by at least 1, as both b_i and b_j are integers. This implies that

$$2^{b_j} \geq 2 \times 2^{b_i},$$

and hence that

$$n + j \geq 2(n + i).$$

On the other hand, the least number in S is $n + 1$, and the largest number is $2n$, and $2n$ is strictly less than twice $(n + 1)$. So it *cannot* happen that $n + j \geq 2(n + i)$ for some pair of numbers $i, j \in \{1, 2, \dots, n\}$.

Therefore it cannot happen that $a_i = a_j$ for some pair $i \neq j$. In other words, the a_i's are all distinct from one another. Where does this leave us? The n odd numbers $a_1, a_2, \dots, a_n$ all lie between 1 and $2n - 1$, and no two are the same. Hence it must be that the string $(a_1, a_2, \dots, a_n)$ is simply a permutation of the string $(1, 3, \dots, 2n-1)$! Therefore the sum of the a_i's is the same as the sum

$$1 + 3 + \cdots + (2n - 1)$$

and we know that this is equal to n^2.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.