A Baby One Quarter the Size of its Parents

C⊗MαC

In Figure 1 we see two 'parent' circles $\mathcal{K}_1$ and $\mathcal{K}_2$ of equal radius tangent to a line ℓ, and a 'baby' circle $\mathcal{K}_3$ tangent to $\mathcal{K}_1$, $\mathcal{K}_2$ and ℓ; the baby has been held tight by its parents! We shall show that the baby has one quarter the radius of its parents. And the main result needed to prove this? It is an old friend, the Pythagorean theorem.

Let the common radius of $\mathcal{K}_1$ and $\mathcal{K}_2$ be taken as 1 unit, and let the radius of $\mathcal{K}_3$ be x units. Let A, B and C denote the centres of $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$, respectively (see Figure 2). Drawing the segments connecting these points, we see that $\triangle ABC$ is isosceles; the base AB has length $1 + 1 = 2$ units, while AC and BC have length $1 + x$ units each. (For, when two circles are

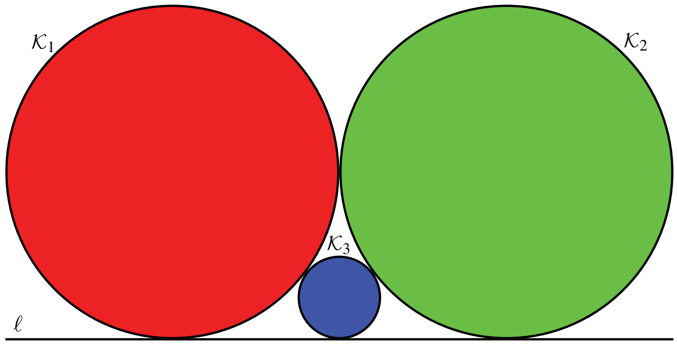

Figure 1.

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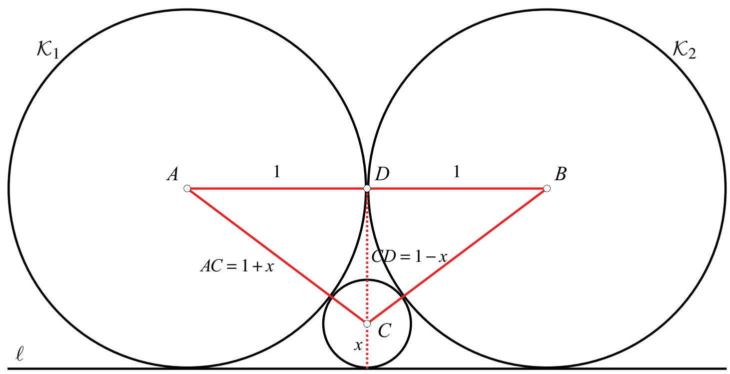

Figure 2.

externally tangent to each other, the distance between their centres equals the sum of their radii.) Segment CD is both a median and an altitude of $\triangle CAB$, and has length $1 - x$ units (because the perpendicular distance of D from ℓ is 1 unit, and the perpendicular distance of C from ℓ is x units).

Now we apply the Pythagorean theorem to $\triangle CAD$, which is right angled at D. We get:

$$\begin{aligned} 1^2 + (1-x)^2 &= (1+x)^2 \\ \therefore\ x^2 - 2x + 2 &= x^2 + 2x + 1, \\ \therefore\ 4x &= 1, \end{aligned}$$

which yields $x = 1/4$. Thus, the baby has 1/4 the radius of the parents, as claimed.

The case of unequal radii

What happens if the two parent circles have unequal radii? Let the parents have radii a and b, respectively (see Figure 3). Denote the radius of the baby circle by c. We shall show that c may be found using the following elegant and symmetric relationship:

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

What we proved above is a particular case of this formula; for if $a = 1 = b$, then the formula gives $1/\sqrt{c} = 2$, so $c = 1/4$.

To prove this we first solve an auxiliary problem (see Figure 4): *What is the length of the tangent segment PQ on ℓ?* We answer this by drawing the segment $BR \parallel PQ$. Then we have: $BR = QP$, $AB = a + b$, $AR = |a - b|$, and now by the theorem of Pythagoras:

$$BR^2 = (a+b)^2 - (|a-b|)^2 = 4ab,$$

which yields $BR = 2\sqrt{ab}$. Therefore, $PQ = 2\sqrt{ab}$.

If we apply this result to the pairs of circles $\{\mathcal{K}_1, \mathcal{K}_3\}$ and $\{\mathcal{K}_2, \mathcal{K}_3\}$, we get:

$$PT = 2\sqrt{ac}, \qquad TQ = 2\sqrt{bc}.$$

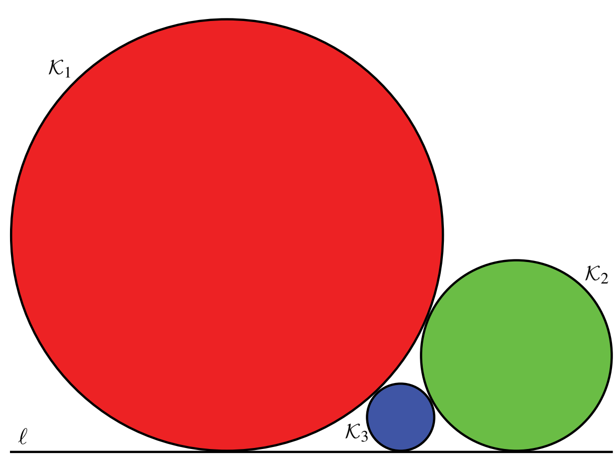

Figure 3.

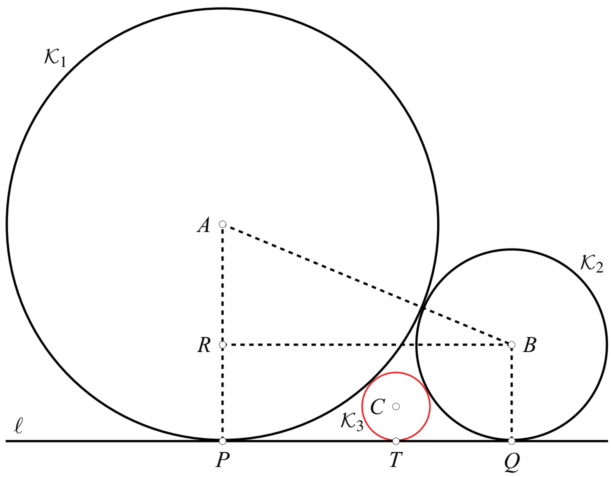

Figure 4.

Since $PT + TQ = PQ$, we obtain:

$$2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}. \qquad (1)$$

On dividing through by $2\sqrt{abc}$ we immediately get the desired relation:

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}. \qquad (2)$$

For example, if $a = 1/4$ and $b = 1/9$, then $c = 1/25$.

One can visualize an unending sequence of circles being constructed in this way:

- a circle $\mathcal{K}_4$ enclosed by $\mathcal{K}_1$, $\mathcal{K}_3$ and ℓ;
- a circle $\mathcal{K}_5$ enclosed by $\mathcal{K}_2$, $\mathcal{K}_3$ and ℓ;
- a circle $\mathcal{K}_6$ enclosed by $\mathcal{K}_3$, $\mathcal{K}_4$ and ℓ;

and so on.

As a special case of formula (2), we have the following:

$$\text{If } a = \frac{1}{m^2} \text{ and } b = \frac{1}{n^2}, \text{ then } c = \frac{1}{(m+n)^2}. \qquad (3)$$

And here is a lovely consequence of (3) for which we invite you to provide the complete justification:

If the radii of the initial two circles $\mathcal{K}_1$ and $\mathcal{K}_2$ are $1/m^2$ and $1/n^2$ for some two integers m and n, then every circle in this infinite chain has a radius of the form $1/p^2$ for some integer p.

Figure 5 shows a few such circles. The configuration makes for colourful pictures!

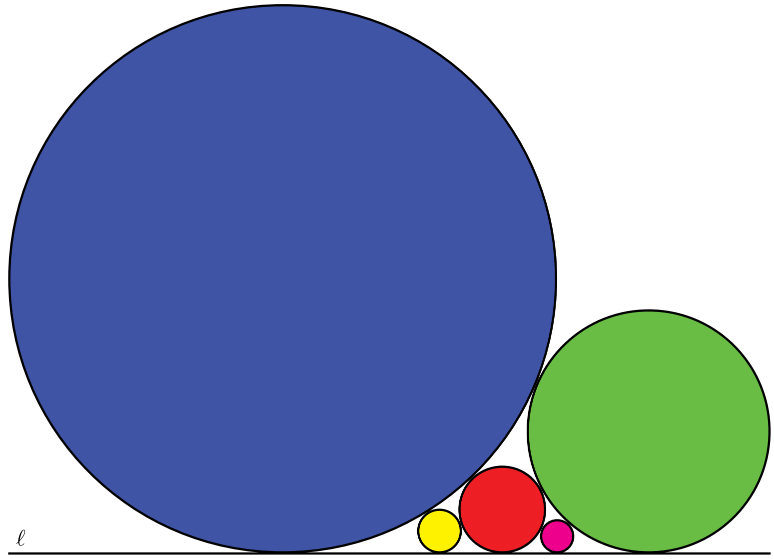

Figure 5. Circles galore

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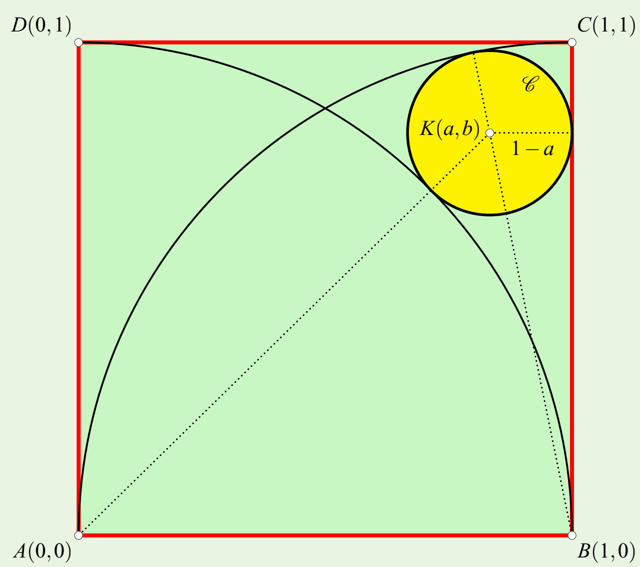
Solution to the "Circle Challenge"



Let the plane be coordinatized so that the coordinates of the vertices A,B,C,D of the square and the centre K of circle $\mathscr{C}$ are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ and (a,b), respectively. Then the perpendicular distance from K to side BC is $1-a$, hence the radius of $\mathscr{C}$ is $1-a$. Now we invoke the result that if two circles touch each other, the distance between their centres is equal to the *sum of their radii* in the case of external contact, and the *difference between their radii* in the case of internal contact. This yields $KA = 1+(1-a) = 2-a$ and $KB = 1-(1-a) = a$. Hence, using the distance formula:

$$a^2 + b^2 = (2-a)^2,$$

$$(a-1)^2 + b^2 = a^2.$$

Subtraction yields: $a^2 - (a-1)^2 = (2-a)^2 - a^2$, giving $2a - 1 = 4 - 4a$, and $6a = 5$. Hence $a = 5/6$. It follows that the radius of the circle is $1/6$.

- Adapted from solution submitted by ***Tejash Patel of Patan, Gujarat.***
A similar solution was sent in by ***Adithya of BGS National Public School.*** *Thanks to both our solvers!*