3, 4, 5 ...

And other memorable triples — Part I

feature

What's interesting about the triple of consecutive integers $3, 4, 5$*? Almost anyone knows the answer to that: we have the beautiful relation* $3^2 + 4^2 = 5^2$*, and therefore, as a consequence of the converse of Pythagoras' theorem, a triangle with sides* $3, 4, 5$ *is right-angled.*

Shailesh Shirali

It is easy to show that $(3, 4, 5)$ is *the only triple of consecutive integers which can serve as the sides of a right-angled triangle*. But in fact rather more can be said, which also makes the matter that much more interesting:

Theorem 1. *Let* $n > 1$ *be an integer. Then the triangle with sides* $n, n+1, n+2$ *is obtuse-angled for* $n = 2$*; right-angled for* $n = 3$*; and acute-angled for all* $n > 3$*.*

The statement is depicted in Figure 1. To see why the claim made in the theorem is true, we examine the expression $n^2 + (n+1)^2 - (n+2)^2 = n^2 - 2n - 3$, which conveniently factorizes as $(n+1)(n-3)$. From this we infer the following:

$$n^2 + (n+1)^2 - (n+2)^2 \text{ is } \begin{cases} < 0 & \text{for } n = 2, \\ = 0 & \text{for } n = 3, \\ > 0 & \text{for } n > 3. \end{cases}$$

The generalized version of Pythagoras' theorem now implies the stated result. (To refresh your memory, here is what this theorem asserts: *In* $\triangle ABC$*, the quantity* $a^2 + b^2 - c^2$ *is greater than, equal to, or less than* 0*, depending on whether* $\measuredangle C$ *is greater than, equal to, or less than a right angle.*)

Keywords: *Pythagoras, triple, acute, obtuse, consecutive integers, touching circles, trisection, in-radius*

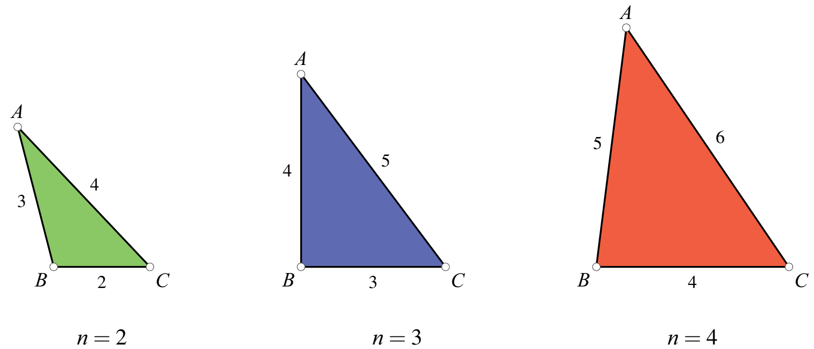

Figure 1. Triangle with sides n, $n + 1$ and $n + 2$

Remark 1. We may also express the above argument in terms of the cosine rule which states that in $\triangle ABC$, the cosine of the angle opposite side a is equal to $(b^2 + c^2 - a^2)/2bc$. Using this we find that in the triangle with sides $n, n+1$, $n+2$, the cosine of the largest angle (which will be opposite the largest side) is:

$$\frac{n^2 + (n+1)^2 - (n+2)^2}{2n(n+1)} = \frac{n-3}{2n}$$

(on simplification).

We see that the cosine of this angle is negative for $n = 2$, zero for $n = 3$, and positive for $n > 3$. The conclusion obtained is the same as earlier: the triangle is obtuse-angled for $n = 2$, right-angled for $n = 3$, and acute-angled for $n > 3$.

Remark 2. The condition $n > 1$ is needed so that the sides satisfy the triangle inequality: "Any two sides of a triangle are together greater than the third one." The inequality fails for $n = 1$, since $1 + 2 = 3$, and we get a 'flat' triangle with angles of 180°, 0° and 0°. (The above formula for the cosine shows that the cosines of the angles are −1, 1 and 1, corresponding to angles of 180°, 0° and 0°.) If the definition of obtuseness can be extended to cover such a triangle, then we do not need to include the condition $n > 1$; we could just say: "Let n be a positive integer."

Thus the triple $(3, 4, 5)$ has some pretty features. We now get a bit greedy and ask: *Are there other nice features that this triple has?* We find that it does, and in this article—which is the first in a multi-part series—we shall describe three such features.

In follow-up articles of the series we will ask: *Are there other triples of consecutive integers which possess geometric features of interest?* This is an open-ended question and many different kinds of results can be envisaged, depending on which "features of interest" we choose to examine. But of that, more later.

Three circles within a circle

In Figure 2 (a), we see a circle $\mathcal{C}_1$ with three circles within it, all tangent to it and also to each other. Two of them, $\mathcal{C}_2$ and $\mathcal{C}_3$, have half the size of $\mathcal{C}_1$ (and therefore pass through the centre O of $\mathcal{C}_1$). The remaining one, $\mathcal{C}_4$, fits tightly in one of the spaces enclosed by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

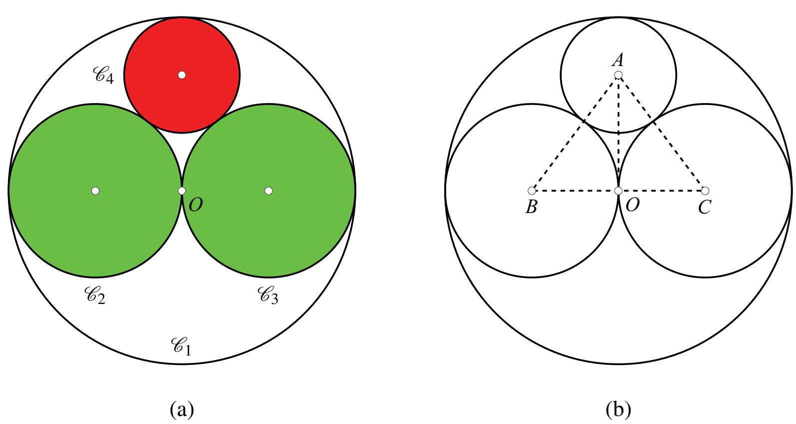

Figure 2. Finding the radius of $\mathcal{C}_4$

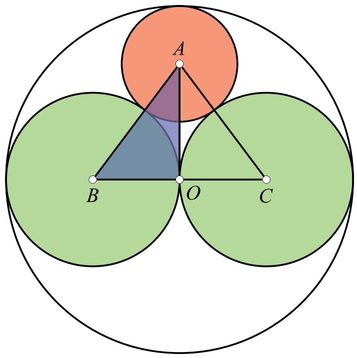

Figure 3. A 3-4-5 triangle hidden within the figure

Problem. To find the radius of $\mathcal{C}_4$.

Let A, B, C be the centres of $\mathcal{C}_4, \mathcal{C}_2, \mathcal{C}_3$, respectively. Note that B, C are collinear with O; A and O are collinear with the point of contact of $\mathcal{C}_1$ and $\mathcal{C}_4$; and A and B are collinear with the point of contact of $\mathcal{C}_2$ and $\mathcal{C}_4$. Let $\mathcal{C}_2$ and $\mathcal{C}_3$ have unit radius, and let the radius of $\mathcal{C}_4$ be x. Then, in Figure 2 (b), $\triangle ABC$ is isosceles with $AB = AC = x + 1$, and $BC = 2$. Also, the altitude $AO = 2 - x$.

Now we are in a position to apply the Pythagorean Theorem to the right-angled $\triangle AOB$; we get:

$$(x+1)^2 = (2-x)^2 + 1^2.$$

This simplifies to $6x = 4$, giving $x = 2/3$. Hence the radius of $\mathcal{C}_4$ is 1/3 that of $\mathcal{C}_1$.

Now let us focus our attention on $\triangle AOB$. Its side lengths are the following:

$$AB = 1 + \frac{2}{3} = \frac{5}{3}, \quad AO = 2 - \frac{2}{3} = \frac{4}{3}, \quad BO = 1,$$

which means that $BO : AO : AB = 3 : 4 : 5$. So, lurking within this figure is a 3-4-5 triangle! We have shown this in a separate figure (Figure 3).

Folding a third

Given a square piece of paper—the kind used in origami—it is easy to produce folds corresponding to fractions such as 1/2, 1/4, 1/8, 3/4, and so on; repeated halving is involved, and nothing more. It is less clear how we can do the same for a fraction like 1/3. It would seem that we have to resort to visual estimation and/or trial-and-error. In the July 2012 issue of *At Right Angles* (the inaugural issue), Shiv Gaur described an elegant iterative procedure that will divide a rectangular strip into five equal parts; a similar method will yield three equal parts. Here we describe a paper folding method that will directly locate a point of trisection of one side of the square.

In Figure 4 (a), we see a square $ABCD$ folded so that vertex D falls upon the midpoint P of AB. The crease of the fold is QR, and the image of C under the fold is S. The point where PS cuts BC is T.

Claim. *T is a point of trisection of BC, with $BT : CT = 2 : 1$, and therefore, $BT/BC = 2/3$, $CT/BC = 1/3$.*

To prove the claim we move to Figure 4 (b). Let $AB = 2$ and $AQ = x$; then $QD = 2 - x$, hence $QP = 2 - x$ (for, QD coincides with QP after the fold). Applying the Pythagorean Theorem to $\triangle APQ$, we get $(2-x)^2 = x^2 + 1^2$, and this yields $x = 3/4$ on solving for x.

Next we use triangle similarity: $\triangle QAP \sim \triangle PBT$ (look at their angles to see why), so $BT : 1 = 1 : x$, giving $BT = 4/3$. Since $BC = 2$, it follows that $BT : BC = 2 : 3$. Therefore, T is a point of trisection of BC.

And now for our bonus. Let us look again at $\triangle QAP$. Its side lengths are: $QA = 3/4$, $AP = 1$ and

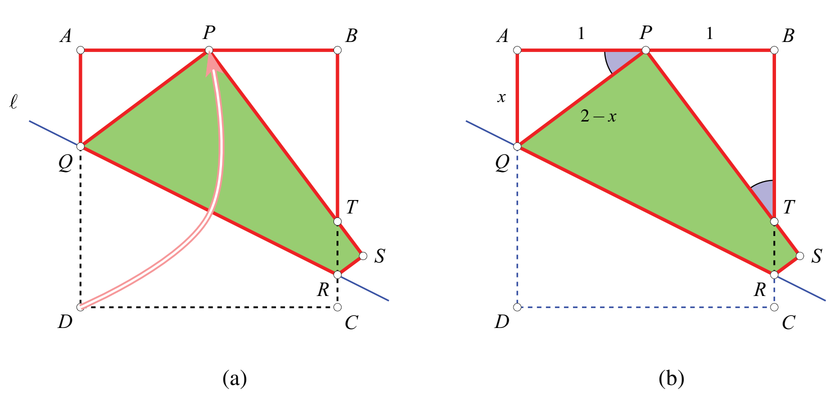

Figure 4.

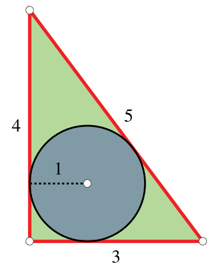

Figure 5. The in-radius of a 3-4-5 triangle is 1 unit

$QP = 2 - 3/4 = 5/4$. Hence its sides are in the ratio $3 : 4 : 5$. And since $\triangle PBT$ is similar to $\triangle QAP$, its sides too are in the ratio $3 : 4 : 5$.

So we have not one but *two* 3-4-5 triangles hidden within this figure.

The in-radius of the 3-4-5 triangle

Our last featured property focuses on what looks like a numerical oddity: the area of the 3-4-5 triangle equals its semi-perimeter. For, its area equals $\frac{1}{2}(3 \times 4) = 6$, and its semi-perimeter equals $\frac{1}{2}(3 + 4 + 5) = 6$. So both have the same value. Those of you who are "physics-minded" may give a cry of outrage here. "This is nonsense! How can area *ever* equal semi-perimeter? Area and semi-perimeter have different dimensions, and one can never equal the other!" That is of course perfectly right, and we shall not make that error here. But the same observation can be translated into a perfectly acceptable form to which no one can object, via this simple formula which connects the in-radius r of a triangle, its area Δ and its semi-perimeter s:

$$rs = \Delta, \quad \text{or} \quad r = \frac{\Delta}{s}.$$

This tells us that for a 3-4-5 triangle, the in-radius is 1 unit. (See Figure 5.) Now we see the source of the dimensionality problem and its resolution at the same time: namely, that the correct relationship is "area equals semi-perimeter times in-radius which equals 1 unit."

The mathematician within us is now provoked to ask the following question: *Are there other integer-sided right-angled triangles whose in-radius is* 1 *unit?* We shall show that the answer is ***No***.

Let ABC be an integer-sided right-angled triangle with $\measuredangle C = 90°$. Let its sides be a, b, c. Then we have:

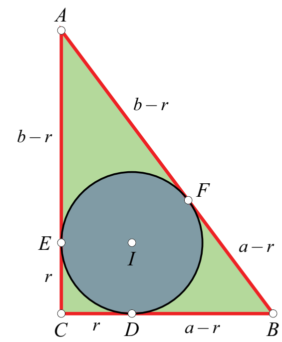

Figure 6. The in-radius of a right triangle: $r = \frac{1}{2}(a + b - c)$

$$\Delta = \frac{ab}{2}, \quad s = \frac{a+b+c}{2}, \quad \therefore r = \frac{\Delta}{s} = \frac{ab}{a+b+c}.$$

Since $r = 1$ we get:

$$ab = a + b + c, \qquad \therefore c = ab - a - b.$$

As the triangle is right-angled, we also have $c^2 = a^2 + b^2$. It follows that

$$(ab - a - b)^2 = a^2 + b^2.$$

We must find pairs of integers that solve the above equation. To avoid duplication of solutions we may assume that $a \le b$. Note that this actually means $a < b$ as we cannot have $a = b$. (We cannot have an integer-sided right-angled triangle which is also isosceles. This is the same as asserting that $\sqrt{2}$ is not a rational number.) Let us now write the above equation as

$$(ab - a - b)^2 - b^2 = a^2.$$

The expression on the left side factorizes as $(ab - a)(ab - a - 2b) = a(b - 1)(ab - a - 2b)$. Hence we have:

$$(b - 1)(ab - a - 2b) = a.$$

Since $a < b$ and a, b are integers, we have $a \le b - 1$. Hence the above equality can hold only if we have $a = b - 1$ and $ab - a - 2b = 1$. These conditions yield:

$$(b-1)b - (b-1) - 2b = 1, \qquad \therefore b^2 - 4b = 0,$$

which yields $b = 4$ (obviously $b \ne 0$) and hence $a = 3$ and $c = 5$. Thus the 3-4-5 triangle is the only integer-sided right-angled triangle whose in-radius is 1 unit.

Alternate solution. Here is another way of reaching the same conclusion. It may be preferred by some, and it also generalizes more easily. It starts by establishing a neat geometrical result: *If $\triangle ABC$ is right-angled with $\measuredangle C = 90°$, then the radius of the incircle of the triangle is* $(a+b-c)/2$. The result has value and interest in itself (mathematicians would call it a 'lemma').

Let the incircle touch the sides BC, CA, AB at points D, E, F respectively. The triangle being right-angled at C, points I, E, C, D form the vertices of a square of side r, hence $CD = r = CE$. From this it follows that $DB = a - r$ and $EA = b - r$. Next, drawing on the fact that the two tangents to a circle from a point outside the circle have equal length, it follows that $AF = b - r$ and $BF = a - r$. From this we get:

$$(a-r)+(b-r)=c, \qquad \therefore\ r=\frac{a+b-c}{2},$$

as claimed.

Now we apply this result to the problem at hand. Let a, b, c be the sides of an integer-sided right-angled triangle whose in-radius is 1 unit. We may assume with no loss of generality that $a < b < c$. The result just proved implies that $a + b - c = 2$, giving $c = a + b - 2$. Invoking the Pythagorean relation we get:

$$a^2+b^2=(a+b-2)^2.$$

This yields: $2ab - 4a - 4b + 4 = 0$, i.e., $ab - 2a - 2b = -2$. Adding 4 to both sides and factorizing, we get:

$$ab-2a-2b+4=2, \qquad \therefore\ (a-2)(b-2)=2.$$

The only way of expressing 2 as a product of two positive integers is $2 = 1 \times 2$, so we must have $a - 2 = 1$ and $b - 2 = 2$ (remember that $a < b$), giving $a = 3$ and $b = 4$ and hence $c = 5$. We reach the same conclusion as earlier.

The advantage of this approach is that it can easily be extended. For example, we may want to list the Pythagorean triples which correspond to triangles with in-radius 2 units. Since the 3-4-5 triangle has in-radius 1 unit, it follows by scaling that the 6-8-10 triangle has in-radius 2 units. Are there any others? Let's see .... Let a, b, c be the sides of an integer-sided right-angled triangle whose in-radius is 2 units; assume that $a < b < c$. Then we have $a + b - c = 4$, giving $c = a + b - 4$. Hence we have:

$$a^2+b^2=(a+b-4)^2.$$

This yields: $2ab - 8a - 8b + 16 = 0$, i.e., $ab - 4a - 4b = -8$. Adding 16 to both sides and factorizing, we get:

$$ab-4a-4b+16=8, \qquad \therefore\ (a-4)(b-4)=8.$$

The ways of expressing 8 as a product of two positive integers are $8 = 1 \times 8 = 2 \times 4$, so the possibilities are:

$$(a-4,b-4)=(1,8), \quad \therefore\ (a,b)=(5,12);$$

$$(a-4,b-4)=(2,4), \quad \therefore\ (a,b)=(6,8).$$

So there are two such triangles—the 5-12-13 triangle and the 6-8-10 triangle—which have in-radius equal to 2 units.

Readers may wish to continue the exploration and search for r-values which give rise to large numbers of candidate triangles.

Closing remark. We have attempted to list some features of the right-angled triangle with sides 3-4-5, and to highlight configurations where this triangle occurs naturally. Without doubt, there are many more such features and many more such configurations. We invite you to design investigations for your students which add to the above list. In the process, students could learn how to make conjectures and then test them and prove them using valid mathematical procedures. May the list grow, and may the conjectures outnumber the theorems!

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