problem corner

Problems for the Senior School

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Problems for Solution

Problem IV-3-S.1
Determine all possible integral values of N such that $N(N-101)$ is the square of a positive integer.

Problem IV-3-S.2
Let R and S be two cubes with sides of lengths r and s, respectively, where r and s are positive integers. Show that the difference of their volumes numerically equals the difference of their surface areas if and only if $r = s$.

Problem IV-3-S.3
Suppose $S = \{0, 1\}$ has the following addition and multiplication rules:

$$0+0=0, \quad 0+1=1+0=1, \quad 1+1=0,$$
$$0\times 0 = 1\times 0 = 0\times 1 = 0, \quad 1\times 1 = 1.$$

(The rules will make more sense if you think of 0 as representing Even, and 1 as representing Odd. For example, the sum of two Odd numbers is Even, and the product of two Odd numbers is Odd.)

A system of polynomials is defined with coefficients in S. The sum and product of two polynomials in the system are the usual sum and product, respectively, where for the addition and multiplication of coefficients the above rules of addition and multiplication apply. For example:

$$\begin{aligned}(x+1)\times\left(x^2+x+1\right) &= x^3+(1+1)x^2+(1+1)x+1\\ &= x^3+0x^2+0x+1 = x^3+1.\end{aligned}$$

Show that in this system x^3+x+1 is not factorisable, that is, one cannot write

$$x^3+x+1 = (ax+b)\times(cx^2+dx+e),$$

where $a, b, c, d, e \in S$.

Problem IV-3-S.4
Consider all non-empty subsets of the set $\{1, 2, 3, \ldots, n\}$. For each such subset, find the product of the reciprocals of each of its elements. Denote the sum of all these products by a_n. For example,

$$\begin{aligned}a_3 &= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1\times 2}+\frac{1}{1\times 3}+\frac{1}{2\times 3}\\ &\quad+\frac{1}{1\times 2\times 3}\\ &= \frac{6+3+2+3+2+1+1}{6} = 3.\end{aligned}$$

Prove that $a_n = n$ for all positive integers n.

Problem IV-3-S.5
Show that the polynomial $x^8-x^7+x^2-x+15$ has no real zero.

Solutions of Problems in Issue-IV-2 (July 2015)

Solution to problem IV-2-S.1
Starting with any three-digit number n we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs and the product of all three digits. (Example: $f(325) = (3+2+5)+(6+15+10)+30 = 71$.) Find all three-digit numbers such that $f(n) = n$. [Adapted from British Mathematical Olympiad, 1994]

Let $n = 100a+10b+c$ where $a, b, c \in \{0, 1, \ldots, 9\}$, $a \neq 0$. Then $f(n) = a+b+c+ab+bc+ca+abc$, so if $f(n) = n$ then we must have

$$99a+9b = ab+bc+ca+abc. \qquad (1)$$

This leads to

$$a(99-b-c-bc) = b(c-9). \qquad (2)$$

Observe that $99-b-c-bc \geq 0$ and $c-9 \leq 0$. Thus equation (2) holds if and only if $c = 9$ and $b+c+bc = 99$, and the latter relation shows that $b = 9$ as well. But when $b = c = 9$, both sides of (1) reduce to $99a+81$, and the equation reduces to an identity. Thus a is not unique but can take any value between 1 and 9. Hence the three-digit numbers n satisfying $f(n) = n$ are 199, 299, 399, 499, 599, 699, 799, 899 and 999.

Solution to problem IV-2-S.2
Solve in integers the equation: $x+y = x^2-xy+y^2$.

The given equation can be written as:

$$(x-y)^2+(x-1)^2+(y-1)^2 = 2. \qquad (3)$$

If $x = y$ then $|x-1| = |y-1| = 1$, whence $(x, y) = (0, 0), (2, 2)$. If $x \neq y$ then $|x-y| = 1$ and either $x = 1$ or $y = 1$. If $x = 1$ then $y = 0$ or $y = 2$. Thus the solutions are $(x, y) = (1, 0), (1, 2)$. Since (y_0, x_0) is also a solution if (x_0, y_0) is a solution, $(x, y) = (0, 1), (2, 1)$ are also integer solutions of the equation. Thus the integer solutions are $(x, y) = (0, 0), (0, 1), (1, 0), (1, 2), (2, 1), (2, 2)$.

Solution to problem IV-2-S.3
Let a, b, c be the lengths of the sides of a scalene triangle and A, B, C be the opposite angles. Prove that

$$\begin{aligned}2(Aa+Bb+Cc) &> Ab+Ac+Ba+Bc\\ &\quad+Ca+Cb.\end{aligned}$$

Without loss of generality, assume that $a > b > c$. Then $A > B > C$. Now:

$$\begin{aligned}&2(Aa+Bb+Cc)\\ &\quad-(Ab+Ac+Ba+Bc+Ca+Cb)\\ &\quad= (A-B)(a-b)+(A-C)(a-c)\\ &\quad\quad+(B-C)(b-c).\end{aligned}$$

Clearly the right-hand side is positive.

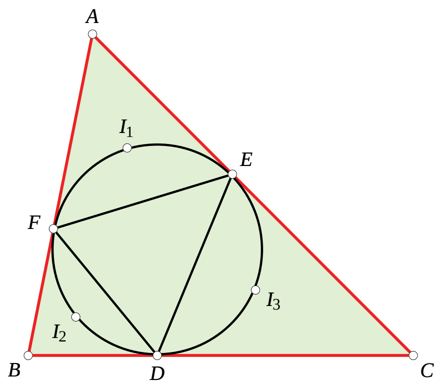
Solution to problem IV-2-S.4
Three positive real numbers a, b, c are such that

$$a^2+5b^2+4c^2-4ab-4bc = 0.$$

Can a, b, c be the lengths of the sides of a triangle? Justify your answer. [Regional Mathematical Olympiad, 2014]

No. Note that $a^2+5b^2+4c^2-4ab-4bc = (a-2b)^2+(b-2c)^2$, hence the stated condition implies that $(a-2b)^2+(b-2c)^2 = 0$, which in turn implies that $a:b:c = 4:2:1$, i.e., $b+c:a = 3:4$. But this violates the triangle inequality, for $b+c$ must exceed a (strictly).

Solution to problem IV-2-S.5
Let D, E, F be the points of contact of the incircle of an acute-angled triangle ABC with the sides BC, CA, AB, respectively. Let I_1, I_2, I_3 be the incentres of the triangles AFE, BDF, CED, respectively. Prove that the lines I_1D, I_2E, I_3F concur. [Adapted from the Regional Mathematical Olympiad, 2014]



Observe that $\measuredangle AFE = \measuredangle AEF = 90^\circ - \frac{1}{2}A$ and $\measuredangle FDE = \measuredangle AEF = 90^\circ - \frac{1}{2}A$. Again, $\measuredangle EI_1F = 90^\circ + \frac{1}{2}A$. Hence:

$$\measuredangle EI_1F + \measuredangle FDE = 180^\circ,$$

so I_1 lies on the incircle. Also:

$$\measuredangle I_1FE = \frac{1}{2}\measuredangle AFE = \frac{1}{2}\measuredangle AEF = \measuredangle I_1EF.$$

Thus $I_1E = I_1F$. But then they are equal chords of a circle, so they subtend equal angles at the circumference. Therefore $\measuredangle I_1DF = \measuredangle I_1DE$ and so I_1D is the internal bisector of $\measuredangle FDE$. Similarly we can show that I_2E and I_3F are internal bisectors of $\measuredangle DEF$ and $\measuredangle DFE$, respectively. Thus the three lines I_1D, I_2E, I_3F are concurrent at the incentre of triangle DEF.

ICMI AWARDS 2015

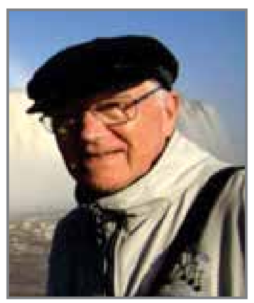
The International Committee for Mathematics Instruction (ICMI) has just announced the recipients of the 2015 Felix Klein and Hans Freudenthal Awards:

- **Felix Klein Medal:** Professor Alan J. Bishop, Emeritus Professor of Education at Monash University, Melbourne.
- **Hans Freudenthal Medal:** Professor Jill Adler, Chair of Mathematics Education, University of the Witwatersrand, South Africa.

The medals will be awarded in the opening ceremony of the International Congress for Mathematics Education (ICME-13) to be held in July 2016 in Hamburg.

Here are brief summaries excerpted from the official citations released by the Awards Committee. See the website http://www.icmihistory.unito.it/ to get a sense of the history of the ICMI and a sense of the work done by Felix Klein and by Hans Freudenthal.

ALAN J. BISHOP



The **Felix Klein medal** is awarded for life-time achievement in mathematics education research. [It] is aimed at acknowledging scholars who have shaped our field over their lifetimes. Past candidates have been influential and have had an impact both at the national level and the international level. [They have] made substantial research contributions, introduced new issues, ideas and perspectives. Additional considerations have included leadership roles, mentoring, and peer recognition.

Alan J. Bishop, Emeritus Professor of Monash University, Australia is the awardee for 2015. He has been instrumental in bringing the political, social, and cultural dimensions of mathematics education to the attention of the field. His early research was on spatial abilities and visualization, but later he worked on the process of mathematical enculturation and how it is carried out in different countries. Subsequently he developed the notion of mathematics as a cultural product. Over more than 45 years of sustained, consistent work, this led to a great deal of work on the political and social dimensions of mathematics education.

Alan Bishop served as editor of *Educational Studies in Mathematics* from 1979 to 1989. In 1980, he founded and became the series editor of Kluwer's *Mathematics Education Library*. He served as the chief editor for many editions of the *International Handbook of Mathematics Education*. Through his tireless work in the area of publication, he enabled research in mathematics education to become an established field.

His education was at Harvard and later at Hull. After a stint at Cambridge University, he moved to Monash University, Australia. Through the Association of Teachers of Mathematics, he worked as a mentor to numerous teachers and supervised many doctoral students, several of whom became distinguished internationally. Through his work in forging links between research and practice, he helped mathematics educators establish communities of inquiry by teaching courses, speaking at conferences and workshops, directing research and development projects, and serving as a consultant, a project evaluator, and an external examiner.

As noted by one of the nominators, "Alan is an excellent scholar and researcher who shaped our field not only over his lifetime but also over its lifetime, not only in England and Australia ... but also internationally."

review

Review of The Sand Reckoner by Gillian Bradshaw[1]

'There was a world there, a world without material existence but luminous with pure reason, and they couldn't see it!'

Dakshayini Suresh

Archimedes (Syracuse, 287 BC-212 BC) is generally believed to have been the greatest mathematician of antiquity, and certainly one of the three greatest of all time (along with Newton and Gauss). He is probably known best for his articulation of what has come to be known as the Archimedes principle, or rather for the entertaining scene that is said to have ensued upon its discovery. The story goes as follows.

Archimedes was asked by King Hieron of Syracuse to determine whether a gold wreath he had commissioned and subsequently received was, in fact, silver. While turning this problem over in his mind, Archimedes chanced to go for a bath, and it struck him, as he bathed, that the volume of water displaced by his being in the bath was equal to the volume of his own body. When he made this discovery, he is said to have run straight out of the bath and his home naked, shouting 'Eureka, eureka!' ('I have found it!'). He used this rule of displacement to determine whether the crown actually was pure and weighed as much as a pure gold object of the same volume.

[1] The title of the novel is the name of a treatise by Archimedes in which he sets out to estimate the number of grains of sand it would take to fill the universe. In the opening scene of the novel, Bradshaw's Archimedes is trying to imagine values large enough to express the number of grains of sand in a box. In the novel, Archimedes also designs a catapult that comes to be known as the Sand Reckoner.

Keywords: *Archimedes, Syracuse, Eureka*