problem corner

Problems for the Middle School

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Problems for Solution

Problem IV-3-M.1
A number when increased by its cube results in the number 592788. Find the number.

Problem IV-3-M.2
Find the two prime factors of 206981 given that one of them is approximately three times the other.

Problem IV-3-M.3
How would you distribute 44 pencils to 10 students such that each student receives a different number of pencils?

Problem IV-3-M.4
Find the sum of all three-digit numbers $\overline{ABC}$ such that the two-digit numbers $\overline{AB}$ and $\overline{BC}$ are both perfect squares.

[Jamaican Math Olympiad 2015]

Problem IV-3-M.5
The numbers $1, 2, 3, 5, 7, 11, 13$ are written on a board. You may erase any two numbers a and b and replace them by the single number $ab + a + b$. After repeating this process several times, only one number remains on the board. What might be this number?

[Adapted from UAB MTS: 2006-2007]

Problem IV-3-M.6
Between 3 PM and 4 PM, Ramya looked at her watch and noticed that the minute hand was between 5 and 6. When she looked next at the watch, slightly less than two hours later, she noticed that the hour and minute hands had switched places. What time was it when she looked at the watch the second time?

[Adapted from "Mathematical Wrinkles" by S.I. Jones, 1912]

Solutions of Problems in Issue-IV-2 (July 2015)

Solution to problem IV-2-M.1
(a) Find the sum of the prime divisors of 2015.

Since $2015 = 5 \times 13 \times 31$, the required sum is $5 + 13 + 31 = 49$.

(b) Find another number for which the sum of the prime divisors is the same.

We only need a collection of primes whose sum is 49. There are many such collections, for example: $7, 19, 23$, giving the number $7 \times 19 \times 23 = 3059$.

Another such collection is $3, 17, 29$, giving the number $3 \times 17 \times 29 = 1479$.

Comment. The statement of the problem allows for some degree of ambiguity if we permit repeated primes in the prime factorisation of the number. For example, what is the sum of the prime divisors of 18? Is it $2 + 3 = 5$, or is it $2 + 3 + 3 = 8$? Obviously, there is no 'is' about the answer; it depends on which interpretation we decide to follow.

If the latter interpretation is followed, then the problem has solutions like $13 + 13 + 23$, giving the number $13 \times 13 \times 23 = 3887$.

Solution to problem IV-2-M.2
The sum of the digits of a natural number n is 2015*. Can n be a perfect square?*

No. To prove this, we use the test for divisibility by 3 ("The remainder that a number leaves on division by 3 is equal to the remainder that its sum of digits leaves on division by 3"), and the fact that on division by 3, every square number leaves remainder 0 or 1 (i.e., no square is of the form $3k + 2$). Now observe that 2015 is of the form $3k + 2$. (Its sum of digits is 8 which is of the form $3k + 2$. Or, more directly: $2015 = 3 \times 671 + 2$.) Hence n too is of the form $3k + 2$. Invoking the fact noted above, we deduce that n is not a perfect square.

Solution to problem IV-2-M.3
Is there a five-digit perfect square such that when 1 *is added to each digit, the answer is again a perfect square? Assume that the addition of* 1 *to each digit starts from the units end and proceeds leftwards, with carry.*

Let a^2 be the five-digit perfect square; then we have $a^2 + 11111 = b^2$ (i.e., another perfect square; of course, $b > a$). This yields: $b^2 - a^2 = 11111 = 41 \times 271$. (Yes, we do need to work out this factorisation!) This yields: $(b - a) \times (b + a) = 1 \times 11111 = 47 \times 271$, hence either $b - a = 1$ and $b + a = 11111$, or $b - a = 41$ and $b + a = 271$. The former possibility yields:

$$b = \frac{1}{2}(11111 + 1) = 5556,$$

$$a = \frac{1}{2}(11111 - 1) = 5555,$$

giving $a^2 = 5555^2 = 30858025$. But this is certainly not a five-digit number. So it cannot be the answer we seek.

The second possibility yields:

$$b = \frac{1}{2}(271 + 41) = 156,$$

$$a = \frac{1}{2}(271 - 41) = 115,$$

giving $a^2 = 115^2 = 13225$. Observe that if we add 1 to each digit of this number, starting from the units end, we get 24336, which equals 156^2. Hence the sought-after answer is: 13225.

Solution to problem IV-2-M.4
The sum of three integers is 0*. Show that the sum of their fourth powers when doubled yields a perfect square.*

Let the three numbers be a, b, c; then $a + b + c = 0$, so $c = -(a + b)$. Hence:

$$\begin{aligned} &2\left(a^4 + b^4 + c^4\right) \\ &\quad = 2\left(a^4 + b^4 + (a + b)^4\right) \\ &\quad = 2\left(2a^4 + 2b^4 + 4a^3b + 6a^2b^2 + 4ab^4\right) \\ &\quad = 4\left(a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4\right) \\ &\quad = 4\left(a^2 + ab + b^2\right)^2. \end{aligned}$$

Hence the sum of the fourth powers when doubled equals the square of $2\left(a^2+ab+b^2\right)$.

Solution to problem IV-2-M.5

Consider these relations:

(1) $a-b-c=0$,
(2) $a^4+b^4+c^4=2\left(b^2c^2+c^2a^2+a^2b^2\right)$.

It is easy to prove (2) from (1) by simple manipulation. Now the interesting thing is: while identity (2) is symmetric in a, b, c, condition (1) is not so. How do you explain this?

The explanation lies in the full factorisation of the expression $P(a,b,c)$ given by:

$$P(a,b,c)=a^4+b^4+c^4-2\left(b^2c^2+c^2a^2+a^2b^2\right).$$

The information given in the problem implies that $a-b-c$ is a factor of $P(a,b,c)$. By symmetry, it follows that the following two expressions are factors as well: $b-c-a$, $c-a-b$. Hence $Q(a,b,c)=(a-b-c)(b-c-a)(c-a-b)$ is a divisor of $P(a,b,c)$.

Now observe the following: if we swap b and c in the expression $Q(a,b,c)$, two factors swap places while the third one remains the same, so the product of the factors remains the same. This tells us that if we were to expand the expression $(a-b-c)(b-c-a)(c-a-b)$, we would get an expression which is symmetric in a, b, c. And indeed we do:

$$(a-b-c)(b-c-a)(c-a-b)=a^3+b^3+c^3 - \left(ab^2+a^2b+bc^2+b^2c+ca^2+c^2a\right),$$

which is symmetric as claimed.

What could be the fourth factor of $P(a,b,c)$? If we write

$$P(a,b,c)=(a-b-c)(b-c-a)(c-a-b) \times \text{a fourth factor},$$

we see (by comparing degrees on both sides) that the fourth factor must be of degree 1. We also see that it must be completely symmetric in a, b, c; for it equals the ratio of two symmetric forms (P and Q) and hence is forced to be symmetric as well. These two conditions imply that it must be of the form $k(a+b+c)$ where k is some constant. By comparing the coefficient of a^4 on both sides, we quickly see that $k=1$. Hence:

$$P(a,b,c)=(a-b-c)(b-c-a)(c-a-b)(a+b+c).$$

So the desired explanation is this: though $a-b-c$ is not symmetric, it has companion factors, and together these factors make for a symmetric expression.

Solution to problem IV-2-M.6

Let a, b be two positive real numbers. Denote their product ab by P, and their sum $a+b$ by S. The following facts are known: if S is a constant, then the maximum value of P is $S^2/4$; and if P is a constant, then the minimum value of S is $2\sqrt{P}$. Use these results to find the maximum and minimum values taken by $x^2/\left(1+x^4\right)$.

Let $y=x^2/(1+x^4)$; then $1/y=x^2+1/x^2$. Since the product P of x^2 and $1/x^2$ is a constant ($P=1$), the sum of the two quantities is least when $x^2=1/x^2$, i.e., $x=\pm 1$, and the least value is 2. This means that the least value of $1/y$ is 2, hence the maximum value of y is $1/2$, taken when $x=\pm 1$.

For the minimum value: it is clear that $y \geq 0$, since only squared expressions and positive signs occur in the expression for y. And the value 0 is realizable, at $x=0$. Hence we have our result: $0 \leq y \leq 1/2$. The lower bound is attained at $x=0$, and the upper bound at $x=1$.

Note. For the sake of completeness, we include a proof of the claim made in the statement of the problem: "If S is a constant, then the maximum value of P is $S^2/4$. If P is a constant, then the minimum value of S is $2\sqrt{P}$." Let the two quantities be a, b, and let $S=a+b$, $P=ab$. Invoking the following simple identity,

$$(a+b)^2-(a-b)^2=4ab,$$

we see that

$$4P=S^2-(a-b)^2, \quad S^2=4P+(a-b)^2.$$

From these relations, it follows that if S is a constant, then P will be largest when $a-b$ is least, i.e., when $a=b$. And if P is a constant, then S will be least when $a-b$ is least, i.e., when $a=b$. On substituting $a=b$ in the two relations, the two claims follow immediately.

Solution to problem IV-2-M.7

Given a parallelogram $ABCD$ and a point P inside the parallelogram such that $\measuredangle APB$ and $\measuredangle CPD$ are supplementary. Show that $\measuredangle PBC=\measuredangle PDC$.

There is an elegant pure geometry proof of the claim; it is illustrated in Figure 1. Translate the entire figure through the vector $\overrightarrow{BA}$; this maps A to A_1, B to A, C to D, D to D_1 and P to P_1. Each segment moves parallel to itself under the mapping. Hence $\measuredangle AP_1D=\measuredangle BPC$.

Now consider the quadrilateral AP_1DP. It is cyclic, since $\measuredangle APD+\measuredangle AP_1D=180°$. Hence $\measuredangle DPP_1=\measuredangle DAP_1$ ("angles in the same segment"). But $\measuredangle DPP_1=\measuredangle PDC$, since $PP_1 \parallel CD$, and $\measuredangle DAP_1=\measuredangle CBP$ by the nature of the translation. Hence $\measuredangle PDC=\measuredangle PBC$, as required.

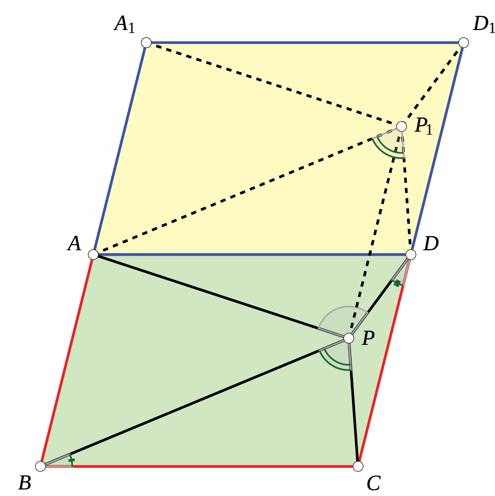

Figure 1.

MAKE THE RELATIONS COME OUT TRUE!

Insert math symbols between/before/after the digits on the left side to make each of the relations true.

- 0 0 0 = 6
- 1 1 1 = 6
- 2 2 2 = 6
- 3 3 3 = 6
- 4 4 4 = 6
- 5 5 5 = 6
- 6 6 6 = 6
- 7 7 7 = 6
- 8 8 8 = 6
- 9 9 9 = 6

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