Problems for the Middle School

Problem Editor : R. ATHMARAMAN

problem corner

Problems for Solution

Problem II-3-M.1

Find the value of the following in as simple a way as possible (and without using a calculator!):

$$\frac{(2013^2 - 2019) \times (2013^2 + 4023) \times 2014}{2010 \times 2012 \times 2015 \times 2016}.$$

Problem II-3-M.2

It is easy to find a pair of perfect squares that differ by 2013; for example, $47^2 - 14^2 = 2013$. Now that the new year (2014) is close upon us, we ask: Can you find a pair of perfect squares that differ by exactly 2014?

Problem II-3-M.3

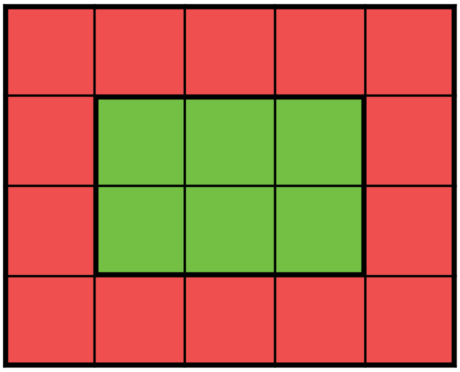
From a two-digit number *n* we subtract the number obtained by reversing its digits. The answer turns out to be a perfect cube. What could be the value of *n*?

Problem II-3-M.4

To a certain two-digit number *m* we add the number obtained by reversing its digits. The answer turns out to be a perfect square. What could be the value of *m*?

Problem II-3-M.5

The rectangle shown has been divided into equal squares. The squares along the perimeter are shaded red, while the rest of the squares are shaded green. You will notice that the number of red squares is greater than the number of green shares. What should be the dimensions of the rectangle if the number of red squares exactly equals the number of green squares?



Solutions of Problems in Issue-II-2

Solution to problem II-2-M.1

Find all integers $n > 0$ such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

This problem has been solved in full in the newly started 'Adventures' column (page 59 − 60).

Solution to problem II-2-M.2

A railway line is divided into 10 sections by stations $A, B, C, D, E, F, G, H, I, J, K$. The distance from A to K is 56 km. A trip along any two successive sections never exceeds 12 km. A trip along any three successive sections is at least 17 km. What is the distance between B and G? (See Figure 1.)

We have: $AD + DG + GJ + JK = 56$. But $AD \geq 17$, $DG \geq 17$, $GJ \geq 17$. Hence $JK \leq 5$.

Again, $HK \geq 17$ and $JK \leq 5$ (just found); hence $HJ \geq 12$. On the other hand, $HJ \leq 12$ (given condition). Hence $HJ = 12$.

Next, since $HK \geq 17$, $HJ = 12$, $JK \leq 5$ we must have $JK = 5$.

In just the same way we get $AB = 5$ and $BD = 12$. Also:
$DH = AK - AB - BD - HJ - JK =$
$56 - 5 - 12 - 12 - 5 = 22$.

Again, $GJ \geq 17$ and $HJ = 12$, hence $GH \geq 5$.

Further, $DG \geq 17$, and $DH = DG + GH = 22$.

Hence $DG = 17$ and $GH = 5$.

Finally we get:

$$BG = BD + DG = 12 + 17 = 29.$$

This is the required answer.

Solution to problem II-2-M.3

In the right $\triangle ABC$ with BC as hypotenuse, $AB = x$ and $AC = y$ where x and y are positive integers. Squares $APQB$, $BRSC$ and $CTUA$ are drawn externally on sides AB, BC and CA, respectively. When QR, ST and UP are joined, a convex hexagon $PQRSTU$ is formed. Let k be its area. Prove that $k \neq 2013$. (See Figure 2.)

Using the sine formula for area of a triangle ("half the product of two sides and the sine of the included angle") and arguing as shown in the figure (in the itemized list), we see that $\triangle ABC$, $\triangle QBR$, $\triangle PAU$ and $\triangle CTS$ have equal area. Hence the total area of the four triangles is $4 \times \frac{1}{2}xy = 2xy$. The areas of the squares are x^2, y^2 and $x^2 + y^2$ (Pythagoras!), so the area of hexagon $PQRSTU$ is $2\left(x^2 + y^2 + xy\right)$ which clearly is even. Hence the area cannot be equal to 2013.

Solution to problem II-2-M.4

The numbers $1, 2, 3, \ldots, n$ are arranged in a line in such a way that each number is either strictly bigger than all the numbers to its left, or strictly smaller than all the numbers to its left. In how many ways can this be done?

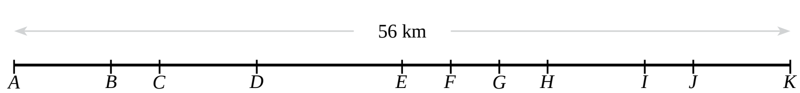
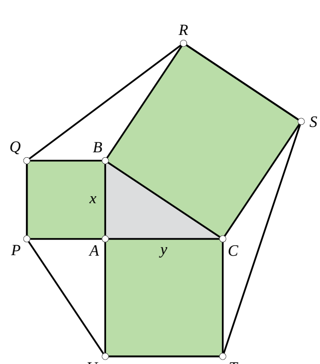


Figure 1.



- $\measuredangle ABC + \measuredangle QBR = 180^0$, hence $\sin\measuredangle ABC = \sin\measuredangle QBR$
- Hence ΔABC and ΔQBR have equal area
- Similarly, ΔABC and ΔPAU have equal area
- Similarly, ΔABC and ΔCTS have equal area

Figure 2.

Let the required number be denoted by $f(n)$. Trivially, $f(1) = 1$. For $n = 2$ both permutations of $(1, 2)$ satisfy the stated condition, so $f(2) = 2$.

For $n = 3$ the permutations of $(1, 2, 3)$ which satisfy the stated condition are $(1, 2, 3)$, $(2, 3, 1)$, $(2, 1, 3)$ and $(3, 2, 1)$; so $f(3) = 4$.

For the general case, with n numbers, let us focus on the *last* number. Since it must be either less than all the other $n - 1$ numbers, or greater than all of them, the last number can only be 1 or n. Whichever it is, the $n - 1$ numbers to its left satisfy exactly the same conditions as the given problem; if the last number is n the numbers are $1, 2, 3, \dots, n - 1$, and if the last number is 1 the numbers are $2, 3, 4, \dots, n$. In the latter case, by subtracting 1 from each number we get a permutation of the numbers $1, 2, 3, \dots, n - 1$ for which the same conditions are satisfied.

From this it follows that for the problem with n numbers there are $f(n - 1)$ permutations in which the last number is n, and an equal number in which the last number is 1. Hence $f(n) = 2f(n - 1)$. Thus the sequence of f-values is a doubling sequence, and since $f(2) = 2 = 2^1$ it follows that $f(n) = 2^{n-1}$.

Solution to problem II-2-M.5

If a, b, c are three real numbers with $1/a + 1/b + 1/c = 1/(a + b + c)$; prove: $1/a^{2n+1} + 1/b^{2n+1} + 1/c^{2n+1} = 1/\left(a^{2n+1} + b^{2n+1} + c^{2n+1}\right)$ for any positive integer n.

The given condition yields, on simplification:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c},$$

$$\therefore (a + b + c)(bc + ca + ab) - abc = 0.$$

The expression $(a + b + c)(bc + ca + ab) - abc$ vanishes if we put $b + c = 0$, or if we put $c + a = 0$, or $a + b = 0$. Invoking the factor theorem, we arrive at this surprising and nice factorization:

$$(a+b+c)(bc+ca+ab)-abc = (a+b)(b+c)(c+a).$$

So the given condition implies that one of $a + b$, $b + c$, $c + a$ is 0. Suppose that $b + c = 0$. Then $b = -c$, so the quantities $1/a^{2n+1} + 1/b^{2n+1} + 1/c^{2n+1}$ and $1/\left(a^{2n+1} + b^{2n+1} + c^{2n+1}\right)$ are both equal to $1/a^{2n+1}$. This is so because $2n + 1$ is an odd integer. Similarly if $a + b = 0$ or $c + a = 0$. Hence proved.