problem corner

Adventures in Problem Solving

# A Tale of Two Formulas

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*Starting with this issue, we will run a new column in the Problem Section, titled 'Adventures in Problem Solving'; it replaces the 'Fun Problems' column we have had till now. In each column we will study a few problems that typify some theme.*

Among the most familiar and humble formulas we meet in high school algebra are the 'completing the square' formula and the 'difference of two squares' factorization. In this note we showcase some unexpected and pleasing uses of these formulas.

## Completing the square

Given the expression $a^2 + 2ab$ we can 'complete the square' by adding the missing last term $b^2$; we get $a^2 + 2ab + b^2 = (a + b)^2$. In Figure 1 we depict this method visually, a depiction we feel should be made better known to students, for its impact as well as its historical connection.

Similarly, if we are given the expression $a^2 + b^2$ we can complete the square by adding the missing middle term which is $2ab$; once again we get $(a + b)^2$. This way of completing the square is much less well known and probably not used to its potential.

Example: Given the expression $n^2 + 10n$, by adding $(10/2)^2 = 25$ to it we get the perfect square: $n^2 + 10n + 25 = (n + 5)^2$; this is what has been depicted in Figure 1. Or, given the expression $n^2 + 25$ we can add $10n$ to it and obtain the same perfect square.

Here is an example showing how 'completing the square' is used at the higher secondary level to solve an integral. Consider the indefinite integral of $1/(x^2 + 10x + 29)$; we have:

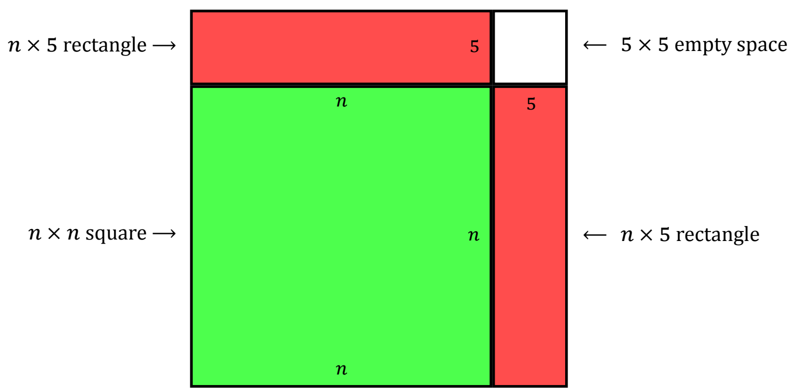
$$\int \frac{dx}{x^2+10x+29} = \int \frac{dx}{(x^2+10x+25)+4}$$

$$= \int \frac{dx}{(x+5)^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right) + \text{constant}.$$

## Difference of two squares

This refers to the identity best remembered in the following form: $a^2 - b^2 = (a - b)(a + b)$. Examples of factorization using the $a^2 - b^2$ formula are (no doubt) familiar to the reader. Here is one involving numbers: $899 = 30^2 - 1 = 29 \times 31$.

There is a lovely geometric interpretation of the act of completing the square which goes all the way back to al Khwarizmi. We illustrate it with the expression $n^2 + 10n$. Here $n^2$ corresponds to the area of a $n \times n$ square, and $10n$ corresponds to the total area of **two** $n \times 5$ rectangles. The three shapes can be fitted together as shown below.




Note the empty space in the upper right-hand corner. Its dimensions are clearly $5 \times 5$. By filling this space, we 'complete the square' . This is the geometric equivalent of adding $5^2$ to $n^2 + 10n$ to get $(n+5)^2$.

Figure 1. *A visual way of 'completing the square'*

And here is an example of its usage in factorizing quadratics. Given the second-degree expression $n^2 + 10n + 16$ we complete the square associated with $n^2 + 10n$, i.e., we add $5^2$ to it and get $(n+5)^2$, and then proceed as follows:

$$n^2 + 10n + 16 = (n+5)^2 - 9 = (n+5)^2 - 3^2 = (n+5+3)(n+5-3) = (n+8)(n+2).$$

In this note we study a few problems that show the two formulas working in unison. We hope to convince you that these simple and unassuming formulas have great power, and we should never underestimate them.

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## Problems

### Problem 1

*Find all integers $n > 0$ such that $n^2 + 12n$ is a perfect square.*

### Solution

Let us first 'complete the square' with the given expression, $n^2 + 12n$. By adding $(12/2)^2 = 36$ we get: $n^2 + 12n + 36 = (n+6)^2$.

Suppose that $n^2 + 12n$ is a perfect square. Let it be denoted by $x^2$. From what was noted above, $n^2 + 12n + 36 = (n+6)^2$ too is a perfect square. Let it be denoted by $y^2$. Here $x$ and $y$ are two positive integers, with $y > x$, and their squares are connected by the relation $y^2 - x^2 = 36$. *So* 36 *has been expressed as a difference of two squares.*

In what ways can 36 be written as a difference of two squares? We find all the ways by calling upon our second friend, the difference of two squares formula! Since a difference of two squares is also a product of two terms, let us first list all the ways of writing 36 as a product of two distinct positive integers. There are several ways: $1 \times 36$, $2 \times 18$, $3 \times 12$, $4 \times 9$.

Since $y^2 - x^2 = 36$ we get $(y-x)(y+x) = 36$; thus, $y - x, y + x$ are two positive integers whose product is 36. Hence it must be that $(y-x, y+x)$ is one of the pairs $(1, 36)$, $(2, 18)$, $(3, 12)$, $(4, 9)$. This information enables us to set up pairs of equations in $x$ and $y$, which can then readily be solved. The results are shown below (though we have left the working out of the solutions to you):

- If $(y-x, y+x) = (1, 36)$, we get $y = 18\frac{1}{2}$, $x = 17\frac{1}{2}$, with $36 = \left(18\frac{1}{2}\right)^2 - \left(17\frac{1}{2}\right)^2$.
- If $(y-x, y+x) = (2, 18)$, we get $y = 10$, $x = 8$, with $36 = 10^2 - 8^2$.
- If $(y-x, y+x) = (3, 12)$, we get $y = 7\frac{1}{2}$, $x = 4\frac{1}{2}$, with $36 = \left(7\frac{1}{2}\right)^2 - \left(4\frac{1}{2}\right)^2$.
- If $(y-x, y+x) = (4, 9)$, we get $y = 6\frac{1}{2}$, $x = 2\frac{1}{2}$, with $36 = \left(6\frac{1}{2}\right)^2 - \left(2\frac{1}{2}\right)^2$.

*In just one case are both $x$ and $y$ are integers.* So only that one possibility works out, and we get: $x = 8$, $y = 10$.

Therefore we get: $(n+6)^2 = 10^2$, hence $n + 6 = \pm 10$. The negative sign yields $n = -16$, which we do not accept, while the positive sign yields $n = 10 - 6 = 4$. *Hence there is precisely one positive integer $n$ for which $n^2 + 12n$ is a perfect square; namely, $n = 4$, for which $n^2 + 12n = 64 = 8^2$.*

## Problem 2

*Factorize the fourth degree polynomial $x^4 + x^2 + 1$.*

## Solution

Since the exponents of the powers of $x$ occurring in the problem are 4 and 2, the given polynomial may be viewed as a quadratic in $x^2$. We may now attempt to complete the square in two ways, by suppressing either the $x^2$ or the 1:

$$x^4 + x^2 = \left(x^4 + x^2 + \frac{1}{4}\right) - \frac{1}{4} = \left(x^2 + \frac{1}{2}\right)^2 - \frac{1}{4},$$

and

$$x^4 + 1 = \left(x^4 + 2x^2 + 1\right) - 2x^2 = \left(x^2 + 1\right)^2 - 2x^2.$$

The first possibility yields:

$$x^4 + x^2 + 1 = \left(x^2 + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

This is a 'sum of two squares' and does not immediately yield results, so we do not pursue it. The other possibility yields:

$$\begin{aligned} x^4 + x^2 + 1 &= \left(x^2 + 1\right)^2 - 2x^2 + x^2 \\ &= \left(x^2 + 1\right)^2 - x^2 \\ &= \left(x^2 + 1 - x\right) \cdot \left(x^2 + 1 + x\right). \end{aligned}$$

Rearranging terms, we get: $x^4 + x^2 + 1 = \left(x^2 - x + 1\right) \cdot \left(x^2 + x + 1\right)$.

## Problem 3

*Show without actually dividing out that* 9901 *is a divisor of* 100010001*.*

## Solution

Since the given claim can be checked by actual division, this cannot be termed a difficult problem! The challenge is to find a solution which is 'pretty and sweet'.

Note that 100010001 may be written as $10^8 + 10^4 + 1$. This means that it is of the form $x^4 + x^2 + 1$ for $x = 10^2$. So we can call upon the result of the preceding problem! We get: $100010001 = \left(10^4 - 10^2 + 1\right) \times \left(10^4 + 10^2 + 1\right) = 9901 \times 10101$.
We see that 9901 is a divisor of the given number. (It turns out that 9901 is prime. The other divisor can itself be factored as $10^4 + 10^2 + 1 = \left(10^2 - 10 + 1\right) \times \left(10^2 + 10 + 1\right) = 91 \times 111$, which then further factorizes, giving $10101 = 3 \times 7 \times 13 \times 37$. So the given number factorizes as
$100010001 = 3 \times 7 \times 13 \times 37 \times 9901$.)

## Problem 4

*Find all integers $n > 0$ such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.*

## Solution

This looks extremely daunting, but we shall tame it using the very same weapons. Our first challenge will be to find an expression which is identically a perfect square and is very close to the given expression, $n^4 - 4n^3 + 22n^2 - 36n + 18$. Since this has degree 4, the desired expression will have to be the square of a quadratic expression. Therefore it must have the form $(n^2 + an + b)^2$ where $a$ and $b$ are coefficients to be found.

Since the coefficient of $n^3$ in the given expression is $-4$, we must have $a = -2$. This is the 'completing the square' rule used yet again! So $n^2 + an + b = n^2 - 2n + b$.

What value should $b$ have so that $(n^2 - 2n + b)^2$ is as close to the given expression as possible? By squaring we get:

$$(n^2 - 2n + b)^2 = n^4 - 4n^3 + (2b + 4)n^2 - 4bn + b^2.$$

So for the coefficients of $n^2$ to 'match' we must have $2b + 4 = 22$, i.e., $b = 9$. But for this value of $b$ we see that the coefficients of $n$ agree as well; only the constant terms differ. Indeed we have:

$$(n^2 - 2n + 9)^2 = n^4 - 4n^3 + 22n^2 - 36n + 81.$$

Now suppose that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square; let it be equal to $x^2$ where $x$ is a positive integer. From what we found above, $n^4 - 4n^3 + 22n^2 - 36n + 81$ too is a perfect square (it is so for any integer $n$); let it be equal to $y^2$, where $y$ is a positive integer.

Looking closely at the two expressions we see that $y^2 - x^2 = 63$. *So 63 has been written as a difference of two squares*. Ah! Now we are on familiar ground.

In what ways can 63 be so written? From the three different factorizations of 63 (as $1 \times 63$, $3 \times 21$, $7 \times 9$) we get all the possible ways by setting up three different sets of equations and solving them:
$63 = 32^2 - 31^2 = 12^2 - 9^2 = 8^2 - 1^2$.

Therefore it follows that $n^2 - 2n + 9$ is one of the numbers 32, 12, 8. Hence to find $n$ we must solve these three separate quadratic equations:

$$n^2 - 2n + 9 = 32,$$
$$n^2 - 2n + 9 = 12,$$
$$n^2 - 2n + 9 = 8.$$

The first of these gives $n = 1 \pm 2\sqrt{6}$ which is not an integer at all (or even rational). The second one yields $n = 3$ or $-1$, while the third one yields $n = 1$.

Since $n$ must be a positive integer we see that $n = 1$ or 3, and we have fully answered the question. The values that the given expression takes at these values of $n$ are $1^2$ and $9^2$.

## Some problems for you to solve, using these ideas

Problem #4 is interesting in that it can be solved in two different ways. Problem #6 is challenging, but do try it out.

1. Factorize these numbers: (i) 3599 (ii) 8099 (iii) 4087.
2. Factorize the polynomial $x^4 + 4$.
3. Find all integers $n$ such that $n^2 + 10n + 20$ is a perfect square.
4. Find all integers $n$ such that $n^2 + n$ is a perfect square.
5. Find all integers $n$ such that $n^2 + n + 1$ is a perfect square.
6. Find all integers $n$ such that $n^4 + n^3 + n^2 + n + 1$ is a perfect square.

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