feature

The ubiquitous triangle

Harmonic Sequence and Pascal's Triangle

An unexpected surprise!

We show in this note how, starting with the infinite harmonic sequence 1, 1/2, 1/3, 1/4, 1/5, 1/6, . . . , *a natural process yields the well-known Pascal triangle and, further, a curious procedure yields back the harmonic sequence. ('Harmonic sequence' is another name for the sequence of reciprocals of the positive integers.)*

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Start with the harmonic sequence arranged in a row of infinite length as

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{7}, \quad \cdots.$$

Subtract each term from the previous term to get the sequence of first differences, with $1 - 1/2 = 1/2$, $1/2 - 1/3 = 1/6$, and so on:

$$\frac{1}{2}, \quad \frac{1}{6}, \quad \frac{1}{12}, \quad \frac{1}{20}, \quad \frac{1}{30}, \quad \frac{1}{42}, \quad \cdots.$$

As we do for the Pascal triangle, write the terms of the second sequence one row below and in-between the terms of the first sequence:

$$\begin{array}{ccccccccccccc} 1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} \\ & \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} & \end{array}$$

Keywords: *Harmonic sequence, Pascal triangle*

Continue in this manner by taking successive differences to get an infinite array:

$$\begin{array}{ccccccccccccc}
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \frac{1}{7} \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{42} & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{105} & & \\
& & & \frac{1}{4} & & \frac{1}{20} & & \frac{1}{60} & & \frac{1}{140} & & & \\
& & & & \cdots & & \cdots & & \cdots & & & &
\end{array}$$

Now turn the above array by 60° (clockwise) to form a triangular array:

$$\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & \frac{1}{2} & & \frac{1}{2} & & & \\
& & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
& \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} \\
& \cdots & & \cdots & & \cdots & & \cdots &
\end{array}$$

Finally, divide each row by the first term in that row:

$$\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & 1 & & 1 & & & \\
& & 1 & & \frac{1}{2} & & 1 & & \\
& 1 & & \frac{1}{3} & & \frac{1}{3} & & 1 & \\
1 & & \frac{1}{4} & & \frac{1}{6} & & \frac{1}{4} & & 1 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

We have obtained a triangle of reciprocals of the Pascal numbers! We call this the *reciprocal Pascal triangle*.

Can we retrieve the harmonic sequence by some natural process? We can. Let us compute the *alternating sums* of the rows of the reciprocal Pascal triangle. Here's what we get: 1, $1-1=0$, $1-1/2+1=3/2$, followed by these numbers:

$$\begin{aligned}
1-\tfrac{1}{3}+\tfrac{1}{3}-1 &= 0,\\
1-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{4}+1 &= \tfrac{5}{3},\\
1-\tfrac{1}{5}+\tfrac{1}{10}-\tfrac{1}{10}+\tfrac{1}{5}-1 &= 0,\\
1-\tfrac{1}{6}+\tfrac{1}{15}-\tfrac{1}{20}+\tfrac{1}{15}-\tfrac{1}{6}+1 &= \tfrac{7}{4},
\end{aligned}$$

and so on. Thus we get the sequence 1, 0, 3/2, 0, 5/3, 0, 7/4, 0, ….

Every second term is 0 (which is not surprising). Deleting these terms we are left with the sequence 1, 3/2, 5/3, 7/4, ... Subtracting each term from 2, we get 1, 1/2, 1/3, 1/4, ... We have got back the harmonic sequence!

Explanation. Inductively, it is easily seen that the r-th row of the array at the top of page 25

$$\frac{1}{r\binom{r}{r}},\quad \frac{1}{r\binom{r+1}{r}},\quad \frac{1}{r\binom{r+2}{r}},\quad \frac{1}{r\binom{r+3}{r}},\quad \cdots\cdots.$$

This follows from the following identity:

$$\frac{1}{r\binom{n}{r}} - \frac{1}{r\binom{n+1}{r}} = \frac{1}{(r+1)\binom{n+1}{r+1}}.$$

After the 60° turn we get array (2); its r-th row is:

$$\frac{1}{r\binom{r}{r}},\quad \frac{1}{(r-1)\binom{r}{r-1}},\quad \frac{1}{(r-2)\binom{r}{r-2}},\quad \frac{1}{(r-3)\binom{r}{r-3}},\quad \cdots,\quad \frac{1}{\binom{r}{1}}.$$

One may rewrite this row as

$$\frac{1}{r\binom{r-1}{r-1}},\quad \frac{1}{r\binom{r-1}{r-2}},\quad \frac{1}{r\binom{r-1}{r-3}},\quad \frac{1}{r\binom{r-1}{r-4}},\quad \cdots,\quad \frac{1}{r\binom{r-1}{0}}.$$

The triangle is thus

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & \frac{1}{2\binom{1}{1}} & & \frac{1}{2\binom{1}{0}} & & & \\
 & & \frac{1}{3\binom{2}{2}} & & \frac{1}{3\binom{2}{1}} & & \frac{1}{3\binom{2}{0}} & & \\
 & \frac{1}{4\binom{3}{3}} & & \frac{1}{4\binom{3}{2}} & & \frac{1}{4\binom{3}{1}} & & \frac{1}{4\binom{3}{0}} & \\
 \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\frac{1}{r\binom{r-1}{r-1}} & \frac{1}{r\binom{r-1}{r-2}} & & \cdots & & \cdots & & \frac{1}{r\binom{r-1}{1}} & \frac{1}{r\binom{r-1}{0}}
\end{array}$$

Dividing each row by the left-most entry (which is the same as the right-most entry), we get the reciprocal of the Pascal triangle, as claimed. Understanding why the alternating sums of the rows give back the harmonic numbers is more intricate, and we refer the reader to the article [1].

References

[1] B.Sury, Tianming Wang & Feng-Zhen Zhao, Identities involving reciprocals of binomial coefficients, Journal of Integer Sequences, Vol.7 (2004), Article 04.2.8.

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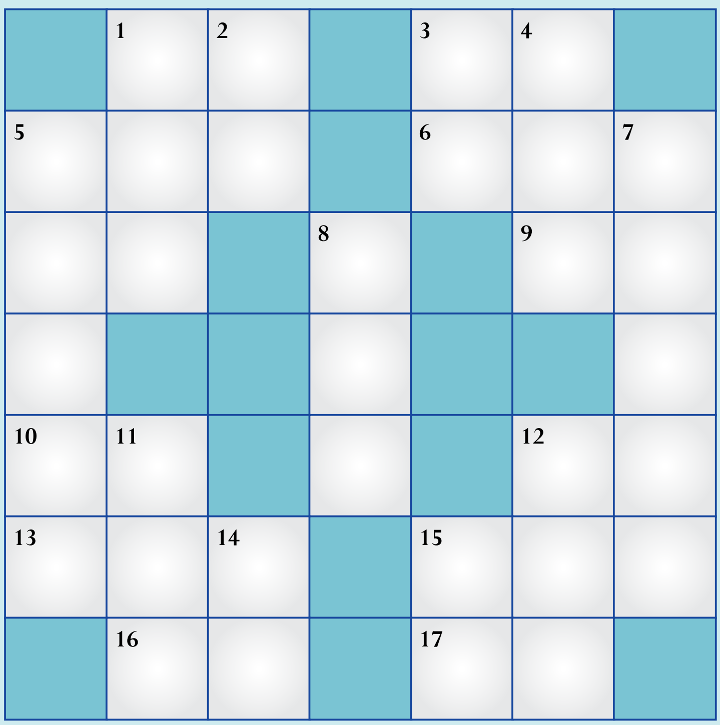
number crossword-4



The design for this crossword's grid as well as several clues were given by Indira Bulhan of class 10 from Techno India Group Public School (Garia). Her interests are astronomy, guitar, dance, singing and drawing.

Down

(1) The reflex angle of 14 degrees

(2) The square of the hypotenuse of a triangle with sides 3 and 5

(3) The arithmetic mean of 2, 15 and 16

(4) 9 more than 1D x 2

(5) 9 less than 2×10^4

(7) 8! – 7! with the digits muddled up

(8) Product of the first four prime numbers

(11) 3A multiplied by the last digit of 15A

(12) Sum of the interior angles of a regular septagon

(14) 16A minus 9A

(15) 3 short of a half century

Clues Across :

(1) Product of first 3 odd numbers subtracted from product of first three even numbers

(3) Half of 2D

(5) A dozen dozens

(6) 3 more than 5 score

(9) Number of right angles in 12D

(10) The sum of the first 13 natural numbers

(12) Five times 10A written in reverse

(13) One of the exterior angles of an isosceles right angled triangle

(15) 3A multiplied by one sixth of 5A

(16) A natural number which is both a perfect square as well as a perfect cube.

(17) One third of 8D