in the classroom

A Puzzle Leading to Some Interesting Mathematics

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A well-known puzzle that can be posed to even young children is as follows:

There is a row of houses numbered sequentially from 1. A resident of one of the houses notices one day that the sum of the door numbers to his left is the same as the sum of the door numbers to his right. How many houses are there in the row and in which of these does the speaker live?

Matters can be made simpler by stating that the number of houses is less than 10. The solution is then not difficult to get by trial and error; see if you can find it.

The next level of challenge is to ask for another solution to the problem. (The numbers involved are still less than 50.) Now trial and error may not be a good option. One could use the well-known formula for the sum of the first n natural numbers

$$S_n = \frac{n(n+1)}{2}$$

to proceed. Let n be the number of houses in the row and m the door number of the house occupied by the speaker. Then the sum of the numbers from 1 to $m-1$ is equal to the sum of the numbers from $m+1$ to n. That is,

$$\frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}, \qquad (1)$$

which leads to

$$m^2 = \frac{n(n+1)}{2}. \qquad (2)$$

The right hand side of the above equation represents a *triangular number*, while the left hand side represents a square number. So we look for numbers that are both square and triangular.

Lists of such numbers are available. The lowest such (apart from 1 itself) is 36: it is the 6^{th} square number and the 8^{th} triangular number. This corresponds to $m = 6$ and $n = 8$, and the door number sums involved are $1 + 2 + 3 + 4 + 5 = 15$ and $7 + 8 = 15$. The next larger values of m and n are, respectively, 35 and 49:

$$1 + 2 + 3 + \cdots + 33 + 34 = 595,$$
$$36 + 37 + 38 + \cdots + 48 + 49 = 595.$$

Though it is not obvious, there are infinitely many pairs (m, n) of whole numbers which satisfy (2). The table below displays some of these pairs:

m	1	6	35	204	1189	6930	...
n	1	8	49	288	1681	9800	...

(3)

As m and n get larger, the ratio n/m gets gradually closer to $\sqrt{2}$. For example:

$$\frac{49}{35} = 1.4, \quad \frac{288}{204} \approx 1.41, \quad \frac{1681}{1189} \approx 1.414, \quad \ldots.$$

We see a slow approach to $\sqrt{2}$. It is not difficult to see why this must be so: we rewrite the equation $m^2 = n(n + 1)/2$ as $n^2 + n = 2m^2$, from which we get:

$$\frac{n^2}{2m^2} = 1 - \frac{n}{2m^2},$$
$$\therefore \frac{n^2}{2m^2} = 1 - \frac{n}{n(n+1)} = 1 - \frac{1}{n+1}.$$

As n gets large, $1/(n + 1)$ gets close to 0, which means that n^2/m^2 gets close to 2.

In (3) we see many striking patterns in both the m-row and the n-row. Example: Write the numbers in the n-row as follows:

$$1^2,\ 8 = 3^2 - 1,\ 7^2,$$
$$288 = 17^2 - 1,\ 1681 = 41^2,\ \ldots. \quad (4)$$

We see that there is an underlying sequence of numbers:

$$1, \quad 3, \quad 7, \quad 17, \quad 41, \quad \ldots, \qquad (5)$$

and we see that this sequence itself has an underlying pattern! Each successive number can be formed using the two previous numbers (much like the rule for the Fibonacci numbers): if a, b, c are three successive numbers in (5), then

$$c = 2b + a. \qquad (6)$$

Example: $41 = (2 \times 17) + 7$. If this pattern is valid, then we ought to be able to guess more numbers in the sequence! After 41 we should have $(2 \times 41) + 17 = 99$, so the next n-value after 1681 should be $99^2 - 1 = 98 \times 100 = 9800$, and it really is so.

Let's continue the pattern. In (5), the number following 99 should be $(2 \times 99) + 70 = 239$, so in (3) the number following 1681 should be $239^2 = 57121$.

Can we now anticipate which number will come after 6930 in the m-row (the top row) of (3)? These numbers too have a simple pattern: if a, b, c are any three consecutive numbers in the row, then

$$c = 6b - a. \qquad (7)$$

Example: $35 = (6 \times 6) - 1$. Therefore after 6930 we should get:

$$(6 \times 6930) - 1189 = 40391.$$

So we expect the following equality to be true:

$$1+2+3+\cdots+40390 = 40392+40393+\cdots+57121.$$

Indeed, it is true: both sides are equal to 815696245. (Please check! Use the fact that the sum of the first n positive integers is $n(n + 1)/2$.)

The reader is invited to find more patterns in (3), and maybe supply some proofs.
(Yes, they *are* needed ....)

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