A Fair Division

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Wills and inheritances have traditionally been a cause of much discord in families. Which is why, when Mr. Jagirdar drew up his will, he invited his friend, the mathematician Mr. Z to meet him at a popular coffee shop and sought his help! You see, the problem was that he wanted to bequeath his triangular plot of land (shown in Fig. 1), bordered on three sides by roads, to his sons, and he wanted to divide the property so that both sons received equal land area *and* equal access to the road. Mr. Z heard out Mr. Jagirdar's problem and looked at the Google map plot of the land shown in figure 1.

Figure 1: Source: http://www.obj.ca/Real-Estate/Construction/2011-11-08/article-2799423/From-homes-to-chips-to-churches/1

Keywords: *Area, perimeter, division, quadratic, discriminant, nature of roots*

Then he looked down at the mouth watering cake that Mr. Jagirdar had ordered for him. Following his gaze, Mr. Jagirdar, exclaimed: *If my sons had to divide the triangle so that each of them got equal cake as well as equal lengths of that delicious chocolate border, then I could have my cake and eat it too!*

Figure 2

But…he added gloomily, If I cut along the median starting from the top to the midpoint of the longest side and give the left portion to my older son, then he is sure to protest that this is not a fair share since the younger fellow has more of that chocolate line (a.k.a. greater access to the road at the right), no matter that the areas are equal.

And if I measure out the chocolate line and cut in such a way that they get equal amounts then they will get unequal areas. I would hate to cause fights simply because I did not make a fair division!

Mr. Z nodded and in true mathematical style, he restated the problem succinctly: *Given a triangle, you want to know if there exists a line which bisects the area as well as the perimeter of the triangle.* Mr. Jagirdar looked abashed at this – was the problem which had troubled him for so long so very easy to state? He had to agree that it was!

Well, said Mr. Z, *if your plot was circular, square or even rectangular, your solution would have been easy.* Yes, said Mr. Jagirdar, *I remember enough about high school geometry to see that for any of these shapes, any line through the centre of the plot would divide it along my specifications.*

Mr. Z smiled and said, Yes; *even if the land had been in the shape of an equilateral or isosceles triangle, your problem would be simpler. Your sons would accept your decision as calmly as when you divided a grilled sandwich for them by cutting it along the median.* Mr. Jagirdar frowned, the conversation was getting too technical for him but Mr. Z had already sketched an equilateral triangle and an isosceles triangle and was briskly explaining that the median to the base of these would also be the perpendicular bisector and voilà, the problem would be solved. Figure 3 shows Mr. Z's sketch.

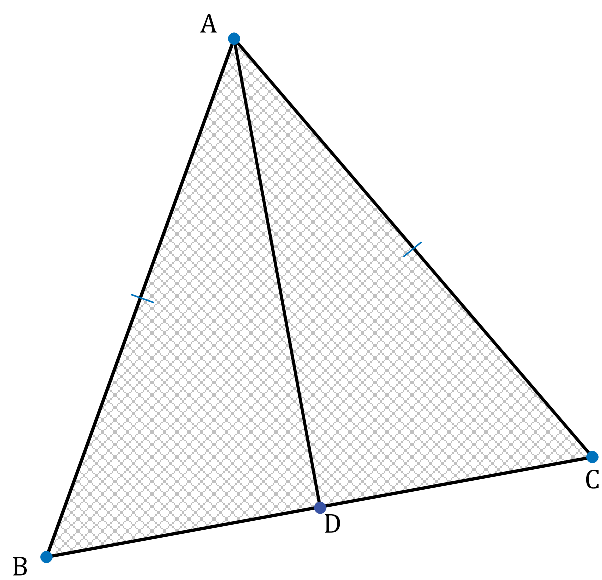


Figure 3

Suppose the given triangle ABC is isosceles and AB = AC. Let D be the midpoint of BC. From fig. 3, we see that AB + BD = AC + CD.

Also, area of triangle *ABD* = ½ *BD* x *AD* = ½ *DC* x *AD* = area of triangle *ADC*

Mr. Jagirdar gloomily stabbed at the cake with his knife. *This scalene triangle makes life difficult! Relax*, said Mr. Z authoritatively. *It is possible, and I will tell you how to make the cut XY so that even the most careful measurements will leave your sons equally satisfied. And I will provide the justification for the division if they want to prove that you have been fair.*

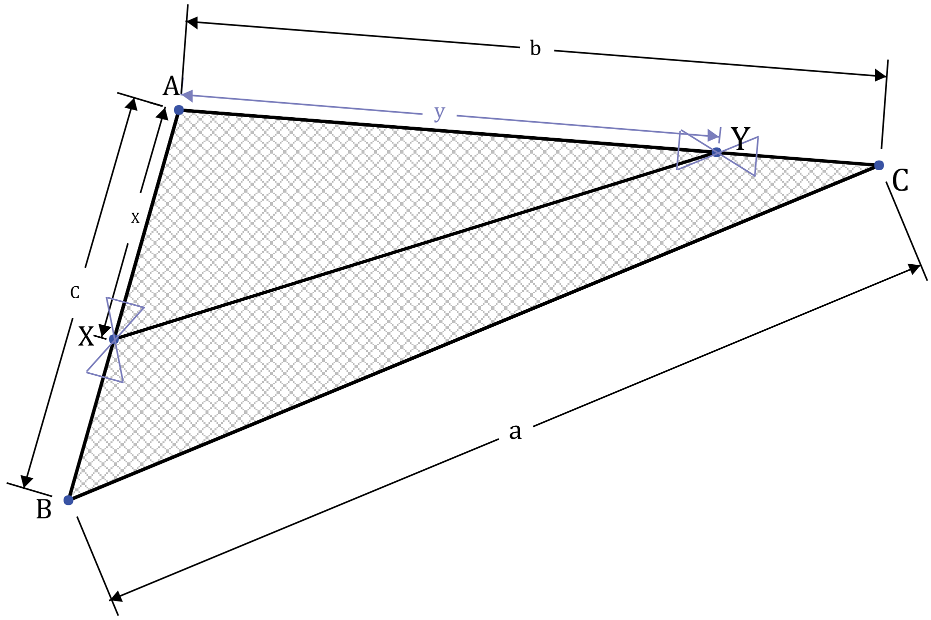


Figure 4

I must see this, said Mr. Jagirdar eagerly. And this is the proof that Mr. Z provided him with:

Let BC = a, CA = b, AB = c and $2s = a + b + c$. Let line L cut the sides AB and AC at X and Y, respectively. Suppose AX = x and AY = y. (See Fig. 4.)

Stage 1: Prove that Area of ΔAXY = ½ Area of ΔABC if and only if $2xy = bc$.

Area of ΔAXY = ½ xy sin A

Area of ΔABC = ½ bc sin A

Substitution in the given expression gives $2xy = bc$. The converse is proved similarly.

Stage 2: Show that XY bisects both the area and the perimeter of the triangle if and only if the simultaneous equations $x + y = s$, $2xy = bc$ have real solutions.

If line XY bisects the area, then $2xy = bc$; and if XY bisects the perimeter of the triangle, then perimeter of triangle AXY = perimeter of quadrilateral XYCB. Substituting the given lengths we get $x + y = s$. So if XY bisects the perimeter and the area, then this set of equations has real solutions. The converse is proved similarly.

Stage 3: Using these simultaneous equations arrive at the condition that the roots are real if $s^2 - 2bc \geq 0$.

Arrive at the quadratic $2x^2 - 2sx + bc = 0$ and find its discriminant.

Stage 4: Prove that $s^2 - 2bc \geq 0$ is equivalent to $(s - b)^2 + (s - c)^2 - (s - a)^2 \geq 0$

Expand $(s - b)^2 + (s - c)^2 - (s - a)^2$. This becomes

$$s^2 + b^2 - 2bs + s^2 + c^2 - 2cs - s^2 - a^2 + 2as$$

$$= s^2 - 2s\,(b + c - a) + (b^2 + c^2 - a^2)$$

$$= s^2 - 2s\,(b + c - a) + ((b + c)^2 - a^2 - 2bc)$$

$$= s^2 - 2s(b + c - a) + (b + c + a)\,(b + c - a) - 2bc$$

On substituting $2s = a + b + c$ we get $s^2 - 2bc$.

Stage 5: Prove that $(s - b)^2 + (s - c)^2 - (s - a)^2 \geq 0$ in turn is equivalent to

$$(s - b)^2 + (a - c)\,b \geq 0.$$

Expand $(s - b)^2 + (s - c)^2 - (s - a)^2$ using the difference of squares formula for the last two terms. The expression becomes

$(s - b)^2 + (s - c - s + a)(s - c + s - a)$. Now substitute $2s = a + b + c$. The expression now reduces to $(s - b)^2 + (a - c)\,b$ which is positive since BC is the longest side

Stage 6: Explain why this expression ensures that the quadratic has real roots.

The discriminant being non-negative ensures that the quadratic has real roots

Stage 7: Prove that the farmer can be successful in his endeavor if he takes

$$x = \frac{s \mp \sqrt{(s^2 - 2bc)}}{2}$$

and $$y = \frac{s \mp \sqrt{(s^2 - 2bc)}}{2}$$

Find the roots of the quadratic equation using the formula.

Awesome said Mr. Jagirdar. *I remember memorizing the conditions for the nature of the roots of a quadratic! I never thought that I would rely them on them to bring peace among my sons!*

That's the problem, said Mr. Z cryptically.

Teacher Note:

Introduction: This problem may seem intimidating at first sight. But it is based on simple principles learnt at school and can be solved by students if they are given a little scaffolding (see blue text for suggested scaffolding and green text for answers which students may be able to provide).

Prior Knowledge:

1. Basic trigonometry

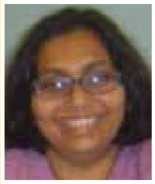
2. Formula for area of a triangle in terms of 2 sides and the included angle.

3. Concept of median of a triangle.

4. The discriminant and the nature of the roots of a quadratic.

Suggestion: Discuss with the students, the seeming contradiction between the diagram which seems to suggest that $x \neq y$ and the result itself which seems to indicate the opposite.

Explain that the positive square root for x coincides with the negative square root for y and vice-versa.



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