Guessing the formula for

Sum of an Arithmetic Progression

in the classroom

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This short note is based on a note written by K. R. S. Sastry (see [1]) in which he puts into practice the constructive pedagogy of George Pólya: "First guess, then prove".

The context used is that of finding a formula for the sum S_n of the first n terms of the arithmetic progression ('AP') with first term a and common difference d:

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \ldots .$$

The textbooks typically give the following formula,

$$S_n = \frac{n\left[2a+(n-1)d\right]}{2},$$

and prove it using Gauss's technique of reversing the terms. As a result, students are rarely if ever presented with the challenge of *finding* a formula (which is clearly not the same as being *given* the formula and then being asked to prove it).

Keywords: *Sequence, consecutive number, generalization, triangular number, pictorial*

But it is comparatively easy to lead students to an answer. We ask them to construct a table of partial sums:

n	Unsimplified sum of first n terms	Simplified sum
1	a	a
2	$a+(a+d)$	$2a+d$
3	$a+(a+d)+(a+2d)$	$3a+3d$
4	$a+(a+d)+(a+2d)+(a+3d)$	$4a+6d$
5	$a+(a+d)+(a+2d)+(a+3d)+(a+4d)$	$5a+10d$

We thus get a sequence of partial sums:

$$a,\ 2a+d,\ 3a+3d,\ 4a+6d,\ 5a+10d,\ 6a+15d,\ 7a+21d, \ldots.$$

Now we must guess a formula for these expressions. The pattern in the sums is easy to see. Each sum is (naturally) a multiple of a added to a multiple of d. The coefficient of a is always equal to the number of terms (again, naturally so). What about the coefficient of d, the numbers 0, 1, 3, 6, 10, . . . ? These numbers should be familiar to students if they have studied the triangular numbers (which surely is a must-study topic at the middle school level), and they may know (or should know!) that the formula $n(n+1)/2$ generates the numbers. In this case, the sequence is displaced by one unit (it starts with 0 rather than 1, the second term is 1 rather than the first term, and so on), hence the formula that applies is $(n-1)n/2$, obtained by replacing n in the previous formula by $n-1$. So it appears by an examination of the expressions that the sum S_n to n terms is given by:

$$S_n = na + \frac{(n-1)n}{2}d.$$

Once the formula has been empirically found, it is easy to see that it must be correct: we get it by adding the 'a' terms and the 'd' terms separately. And it is easy to transform the formula to the usual forms (where 'last term' means 'n^{th} term'):

$$\begin{aligned} S_n &= na + \frac{(n-1)n}{2}d = \frac{2na+(n-1)nd}{2} \\ &= \frac{n\left(2a+(n-1)d\right)}{2} = \frac{n\left[a+\left(a+(n-1)d\right)\right]}{2} \\ &= \text{number of terms} \times \frac{\text{first term} + \text{last term}}{2} \\ &= \text{number of terms} \times \text{average of first term and last term.} \end{aligned}$$

References

i. K. R. S. Sastry, "First guess, then prove", The Mathematics Teacher (pub: National Council of Teachers of Mathematics), Vol. 73, No. 4 (April 1980), pp. 247