feature

PHTs . . . Primitive and beautiful

Harmonic Triples

Part 3

Observe a relationship, then prove it – satisfying in itself.
Take this one step further and find the geometrical connect.
Excitement squared!

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In Parts–1, 2 of this article which appeared in the March 2013 and July 2013 issues of *At Right Angles*, we introduced the notion of a *primitive harmonic triple* ('PHT') as a triple (a, b, c) of coprime positive integers satisfying the equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}.$$

Examples: (3,6,2) and (6,30,5). We showed that the equation surfaces in many contexts, and we explored ways of generating PHTs. Now we explore some properties of these triples. Table 1 lists many of the triples. (To avoid duplication we have added the condition $a \le b$.) It is worth studying the list to identify features of interest.

(2,2,1),	(3,6,2),	(4,12,3),	(5,20,4),
(6,30,5),	(7,42,6),	(8,56,7),	(9,72,8),
(10,15,6),	(10,90,9),	(14,35,10),	(18,63,14),
(21,28,12),	(22,99,18),	(24,40,15),	(30,70,21),
(33,88,24),	(36,45,20),	(44,77,28),	(55,66,30),
(60,84,35),	(65,104,40),	(78,91,42),	(105,120,56).

Table 1. A list of some primitive harmonic triples (PHTs)

Keywords: *Primitive harmonic triples, perfect squares, rhombus, triangle*

Perfect squares

Among the many noticeable features of PHTs, the one that strikes the eye most is the presence of numerous perfect squares associated with each triple.

Proposition 1 *If* (a, b, c) *is a primitive harmonic triple, then* $a + b$, $a - c$ *and* $b - c$ *are perfect squares.*

> Example: Take the PHT $(10, 15, 6)$; we have: $10 + 15 = 5^2$, $10 - 6 = 2^2$, $15 - 6 = 3^2$. But still more can be said.

Proposition 2 *If* (a, b, c) *is a primitive harmonic triple, then* abc *is a perfect square.*

> Example: Take the PHT $(10, 15, 6)$; we have: $10 \times 15 \times 6 = 900 = 30^2$.

Four perfect squares associated with each primitive harmonic triple! Remarkable. But the claims are easier to prove than one may expect. To do so we use the complementary factor algorithm obtained in Part–2 for finding such triples.

The algorithm recalled

For readers' convenience we derive the algorithm afresh. Suppose that a, b, c are positive integers such that $a \le b$ and $1/a + 1/b = 1/c$. By clearing fractions we get:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}, \quad \therefore \frac{a+b}{ab} = \frac{1}{c}, \quad \therefore ac + bc = ab,$$

hence $ab - ac - bc = 0$. Adding c^2 to both sides we get:

$$ab - ac - bc + c^2 = c^2, \quad \therefore (a - c)(b - c) = c^2.$$

So $a - c$ and $b - c$ are a pair of divisors of c^2 whose product is c^2. (Thus, they are a pair of 'complementary divisors' of c^2.) So the algorithm to generate harmonic triples is:

1. Select a positive integer c.
2. Write c^2 as a product $u \times v$ of two positive integers, where $u \le v$.
3. Let $a = c + u$ and $b = c + v$.
4. Then $(a, b, c) = (c + u, c + v, c)$ is a harmonic triple in which $a \le b$.

In Part–2 of the article (July 2013 issue of *At Right Angles*) we remarked that to ensure that a, b, c are coprime (i.e., ensure that the triple is primitive), it is necessary as well as sufficient that u and v be coprime. Now we establish this claim.

Suppose that $(a, b, c) = (c + u, c + v, c)$ is *not* primitive; then there exists a number $k > 1$ which divides each number in the triple. Since k divides both $c + u$ and c, it divides u. Since k divides both $c + v$ and c, it divides v. Therefore, k divides both u and v. Hence u and v are not coprime. Taking the contrapositive of this finding, we deduce that if u and v are coprime, then the triple $(c + u, c + v, c)$ is primitive.

Next we must show the converse: if u and v are not coprime then $(c + u, c + v, c)$ is not primitive. Let p be a prime number which divides u as well as v. Then p^2 is a divisor of c^2, since $uv = c^2$. Since p is prime, it follows that p divides c. Hence p divides each number in the triple $(c + u, c + v, c)$. Consequently the triple is not primitive.

Proof of Proposition 1

Let (a, b, c) be a PHT; then there exist coprime positive integers u and v such that $uv = c^2$, $a = c + u$, $b = c + v$. Since u and v are coprime and their product is a perfect square, each of them must be a perfect square, say $u = r^2$ and $v = s^2$; this yields $a - c = r^2$ and $b - c = s^2$, showing directly that both $a - c$ and $b - c$ are perfect squares, as claimed. Now consider $a + b$. Since $a = c + r^2$ and $b = c + s^2$ and $rs = c$, we have:

$$a + b = c + r^2 + c + \frac{c^2}{r^2} = r^2 + 2c + \frac{c^2}{r^2}$$
$$= \left(r + \frac{c}{r}\right)^2 = (r + s)^2.$$

So $a + b$ too is a perfect square.

Another pretty relation

We have shown that if (a, b, c) is a PHT, then

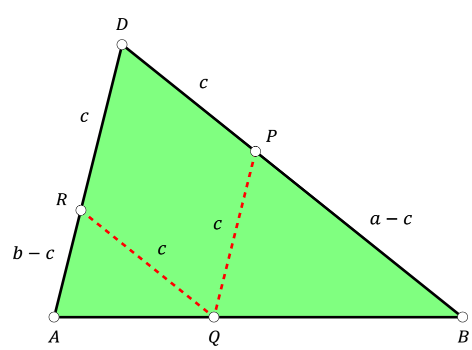
$$a + b = \left(\sqrt{a - c} + \sqrt{b - c}\right)^2.$$

Example: Take the PHT $(10, 15, 6)$; we have: $10 + 15 = 25 = \left(\sqrt{10 - 6} + \sqrt{15 - 6}\right)^2$.

Geometric interpretation of an algebraic relation

We now give a geometric interpretation to the following fact: If (a, b, c) is a harmonic triple then $(a - c)(b - c) = c^2$.

Given a $\triangle ADB$, let a rhombus $DPQR$ be inscribed in the triangle, with P on DB, Q on AB, and R on DA. Figure 1 shows the completed picture. Let a, b, c be the lengths indicated ($a = DB$, $b = DA$, $c =$ side of the rhombus). We had shown earlier that $1/a + 1/b = 1/c$.



- $DB = a$
- $DA = b$
- DP, PQ, QR, RD all have length c

Figure 1: *Rhombus inscribed in a triangle*

Now note the similarity $\triangle BPQ \sim \triangle QRA$, which follows from the relations $PQ \parallel DA$ and $RQ \parallel DB$. From this we get the following relation:

$$\frac{a - c}{c} = \frac{c}{b - c}.$$

By cross-multiplying we get the desired relation: $(a - c)(b - c) = c^2$.

More propositions

Proposition 3 *If* (a, b, c) *is a PHT then* $a + b + c$ *is coprime to* 6*.*

Proposition 4 *If* (a, b, c) *is a PHT then* $a + b - c$ *is coprime to* 10*.*

Proposition 5 *If* (a, b, c) *is a PHT then* $\gcd(a, b) = \gcd(a, c) + \gcd(b, c)$*.*

Example: Consider the PHT $(10, 15, 6)$. Observe that:

- $10 + 15 + 6 = 31$ is coprime to 6;
- $10 + 15 - 6 = 19$ is coprime to 10;
- $\gcd(10, 15) = 5 = 2 + 3 = \gcd(10, 6) + \gcd(15, 6)$.

Proof of Proposition 3

- Let $a = c + r^2$, $b = c + s^2$ where r and s are coprime, and $rs = c$. Then:

  $$a + b + c = 3c + r^2 + s^2.$$

  Since r and s are coprime, it cannot be that both are divisible by 3. So at most one of r, s is a multiple of 3.

- Recall that if n is not a multiple of 3, then $n^2 \equiv 1 \pmod 3$.

  [We do not really need to `recall' it, for it has a one-line proof. We only need to check that $1^2 \equiv 1 \pmod 3$ and $2^2 \equiv 1 \pmod 3$.]

- Suppose that just one of r, s is a multiple of 3; assume it is r. Then $r^2 \equiv 0 \pmod 3$ and $s^2 \equiv 1 \pmod 3$, therefore $r^2 + s^2 \equiv 1 \pmod 3$, and $a + b + c \equiv 1 \pmod 3$.

- If both r and s are non-multiples of 3, then $r^2 \equiv 1 \pmod 3$ and $s^2 \equiv 1 \pmod 3$, hence $r^2 + s^2 \equiv 2 \pmod 3$, and $a + b + c \equiv 2 \pmod 3$.

- So in both cases $a + b + c$ is a non-multiple of 3.

- Now we must show that $a + b + c$ is odd. We do so by focusing on the parity of c, i.e., its odd/even nature.

- If c is odd, then r and s are both odd, hence $a + b + c = 3c + r^2 + s^2$ is the sum of three odd numbers and so is odd. If c is even, then since $rs = c$ and r and s are coprime, it must be that one of $\{r, s\}$ is even and the other is odd. So $a + b + c$ is the sum of two even numbers and one odd number, and hence is odd.

  So in all cases, $a + b + c$ is odd and a non-multiple of 3, and so is coprime to 6.

Proof of Proposition 5

Since the general PHT (a, b, c) has $c = rs$, $a = c + r^2$, $b = c + s^2$ where $\gcd(r, s) = 1$, we must prove the following:

$$\gcd(rs + r^2, rs + s^2) = \gcd(rs + r^2, rs) + \gcd(rs + s^2, rs).$$

But this is clear, because:

$$\gcd(rs + r^2, rs + s^2) = \gcd(r(s + r), s(s + r)) = s + r,$$

$$\gcd(rs + r^2, rs) = \gcd(r(r + s), rs) = r,$$

$$\gcd(rs + s^2, rs) = \gcd(s(s + r), sr) = s.$$

Proofs of Propositions 2 and 4

We leave these for you. Readers must do some work, too!
There are yet more such properties that you may want to explore. For example:

Proposition 6 *If* (a, b, c) *is a PHT, then* $a + b + c$ *is not divisible by* 7.

Proposition 7 *If* (a, b, c) *is a PHT, then* $a + b - c$ *is not divisible by* 11.

See if you can spot (and prove) more such properties!

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Solution for the Number Crossword in Issue-II-3 (November 2013)

	[1] 3	[2] 3		[3] 1	[4] 7	
[5] 1	4	4		[6] 1	0	[7] 3
9	6		[8] 2		[9] 1	0
9			1			2
[10] 9	[11] 1		0		[12] 9	5
[13] 1	3	[14] 5		[15] 4	0	8
	[16] 6	4		[17] 7	0	