Musing on the primes

There are Infinitely Many Primes – II

But how many proofs of this?

feature

In Part–I of this article we dwelt on various proofs that show the infinitude of the primes. These were mostly based on Euclid's proof — the one for which G H Hardy had such high praise. All of these start by assuming that there exists a 'last prime'. Then in a clever way they construct a number whose prime factors exceed this last prime. The one proof discussed which does not belong to this category is Pólya's; he makes use of the Fermat numbers. The first proof of a completely different nature is Euler's; he shows that the sum of the reciprocals of the primes is infinite, and hence there must exist infinitely many primes. In Part–II we dwell on this proof.

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Convergent and divergent series

Say you have a set S of numbers. You want to know whether there are finitely many elements in S, or infinitely many. How may we do this? Here is a possible strategy: *Add up all the numbers in* S. If the sum if infinite, then surely S must have infinitely many elements!

Note that this strategy works only in one direction: If the sum is infinite, then S has infinitely many elements. But if the sum is finite, we cannot say anything about the size of S. This strange situation at one time in history looked impossible, and all kinds of paradoxes arose because of that, like Zeno's paradox. But it is easy

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to show that one *can* add infinitely many numbers and reach a finite sum. The simplest example of this is the following infinite decimal:

$$x = 0.1111111\ldots,$$

i.e., the recurring decimal made up of 1s. It is clearly a sum of infinitely many numbers:

$$x = 0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + 0.000001 + 0.0000001 + \cdots.$$

It is easy to show that $x = 1/9$ (to see this, work out $1/9$ in decimal form using long-division; or multiply the above relation by 10 and then subtract the original relation). So here we have a case where infinitely many positive quantities when added yield a finite number.

The above is of course a special case of the general result:

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}, \quad \text{valid for all real } x \text{ with } |x| < 1. \tag{1}$$

Two nice special cases of this result are:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 1 \quad (\text{with } x = \tfrac{1}{2}),$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \frac{1}{2} \quad (\text{with } x = \tfrac{1}{3}).$$

What about cases where the sum is infinite? An instance which is quite uninteresting is:

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots = \infty.$$

(This is (1) with $x = 1$.) We note an important point here. When we write "$\cdots = \infty$"it is not as though ∞ is a number, like 1 or 2. The phrase "$\cdots = \infty$"is merely a short form to mean that the sum in question has no bound; *by adding a sufficient number of terms, we can get the sum to exceed any given bound*.

Historically, the first result in this area which has genuine surprise value is this:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty, \quad \text{i.e.,} \sum_{1}^{\infty} \frac{1}{n} = \infty. \tag{2}$$

This result is known as 'divergence of the harmonic series'. (The numbers $1, 1 + 1/2, 1 + 1/2 + 1/3, \ldots$are called the 'harmonic numbers', and the series $1 + 1/2 + 1/3 + \cdots$ is called the 'harmonic series'.) The proof (first given by Nicolo Oresme) is a standard result in the subject called 'Analysis'. Why do we say that the result has surprise value? *Because it is counter-intuitive*. If we introduce the harmonic series to students in (say) class 10 or 11, most of them would venture to guess that the series adds up to a finite number. The numerical evidence does suggest this: the first 1000 terms yield a sum of just 7.48, and the following terms appear to not add very much. (For the case of completeness we do give a proof of divergence of the harmonic series in the Appendix.)

What Euler proved is far more counter-intuitive —he showed that *the sum of the reciprocals of the primes is divergent*:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty, \quad \text{i.e.,} \sum_{1}^{\infty} \frac{1}{p_n} = \infty, \tag{3}$$

where p_n is the n^{th} prime. This shows right away that there are infinitely many primes; but it proves much more. Since the corresponding sum for the powers of 2 is finite (as noted above), and the same is true for the perfect squares, that is,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \text{a finite number},$$

it means that in some sense the primes are more 'dense' than either the powers of 2 or the perfect squares. We now give a brief sketch of Euler's proof.

Fundamental theorem of arithmetic (FTA)

The number 60 can be written as a product of prime powers thus: $60 = 2^2 \times 3 \times 5$. Is there any other way of writing 60 as such a product? No. How about $1001 = 7 \times 11 \times 13$? Can it be written as a product of prime powers in any other way? No again. (We do not count $11 \times 7 \times 13$ as different from $7 \times 11 \times 13$.) Both these are instances of the Fundamental Theorem of Arithmetic (FTA), a crucial theorem of number theory; and Euler's proof uses the FTA in a basic way.

Here is the statement of the FTA: *Every positive integer greater than* 1 *can be expressed in precisely one way as a product of powers of prime numbers*. The FTA is often taken by students to be 'obviously' true, and the proof is omitted. But there is need for a formal proof. Interested readers should look up reference [1] (pages 3 and 21) or reference [2] (page 23). Crucial to the proof is the following property of prime numbers: *If p is a prime number, and p divides the product ab of two integers a and b, then p divides a or b or both*. Using this property and the principle of induction, a proof for the FTA may be devised.

Euler's observation

In (eq:1) substitute $x = 1/p$ where p is a prime number; we get:

$$\frac{1}{1-1/p} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \cdots. \tag{4}$$

Next put $x = 1/q$ where q is a prime number different from p; we get:

$$\frac{1}{1-1/q} = 1 + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} + \frac{1}{q^4} + \cdots. \tag{5}$$

Now multiply the corresponding sides of (eq:4) and (eq:5). On the left side we get:

$$\frac{1}{1-1/p} \times \frac{1}{1-1/q}.$$

On the right side we multiply together two infinite series and get another such series:

$$1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{p^2} + \frac{1}{pq} + \frac{1}{q^2} + \frac{1}{p^3} + \frac{1}{p^2q} + \frac{1}{pq^2} + \frac{1}{q^3} + \cdots.$$

The important thing here is that on the right side we get the reciprocals of all integers of the form $p^i q^j$ where i, j are non-negative integers. *Each such integer occurs just once in the above equality*. It is here that FTA plays its role.

Example: The case $p = 2$ and $q = 5$ yields the following equality, since $1/(1-1/2) \times 1/(1-1/5) = 5/2$, and the positive integers with no prime factors other than 2 and 5 are 1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80, 100, ...:

$$\frac{5}{2} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \frac{1}{32} + \frac{1}{40} + \frac{1}{50} + \frac{1}{64} + \frac{1}{80} + \frac{1}{100} + \cdots.$$

If we consider a third prime r, we get the following equality:

$$\frac{1}{1-1/p} \times \frac{1}{1-1/q} \times \frac{1}{1-1/r} = 1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{pq} + \frac{1}{pr} + \frac{1}{qr} + \frac{1}{p^2} + \cdots,$$

and now on the right side we have the reciprocals of all integers of the form $p^i q^j r^k$, i.e., all positive integers which have no prime factors other than p, q, r. So if $p, q, r, \ldots$are distinct prime numbers, then:

- The sum of the reciprocals of all positive integers divisible by no prime number other than p is $1/(1-1/p)$.
- The sum of the reciprocals of all positive integers divisible by no prime numbers other than p, q is $1/(1-1/p) \times 1/(1-1/q)$.
- The sum of the reciprocals of all positive integers divisible by no prime numbers other than p, q, r is $1/(1-1/p) \times 1/(1-1/q) \times 1/(1-1/r)$.

And so on. It was Euler who first thought along such lines, and this led him to say to himself, "Why not list the corresponding relation involving *all* the prime numbers?" Then on the right side we get the reciprocals of *all* the positive integers, each occurring just once. Euler thus wrote:

$$\frac{1}{1-1/2} \times \frac{1}{1-1/3} \times \frac{1}{1-1/5} \times \frac{1}{1-1/7} \times \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots. \tag{6}$$

This is just how Euler wrote the relation. (On the left side, 2, 3, 5, 7, ... are the primes, in sequence, and on the right side, 1, 2, 3, 4, 5, 6, ... are the positive integers, in sequence.) Today, with standards of rigour having changed over the centuries, we do not write it in quite this way, since neither side of (**6**) is a finite number! However in this article we will go with Euler and write the relation in his style.

Euler's proof

The rest of the proof now writes itself. Suppose that there are only finitely many primes $p_1, p_2, \ldots, p_n$; that is, there are just n primes, the largest one being p_n. Then statement (**6**) reads:

$$\frac{1}{1-1/p_1} \times \frac{1}{1-1/p_2} \times \cdots \times \frac{1}{1-1/p_n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots. \tag{7}$$

The left side of this equality is obviously a finite number, being the product of finitely many fractions. But the right side is the harmonic series, and we know that this series diverges; so the right side is infinitely large! It follows that statement (**7**) is an absurdity.

Tracing backwards we see that the contradiction arises from the supposition that the number of primes is finite. We conclude that there are infinitely many primes.

Exercises

1. Show that the sum of the reciprocals of all positive integers with no prime factors other than 2 and 3 (i.e., the integers 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, ...) is 3.
2. Find the sum of the reciprocals of all positive integers with no prime factors other than 3 and 5, i.e., the integers 1, 3, 5, 9, 15, 25, 27, 45, 75, ...

Appendix: Divergence of the harmonic series

Here is one way of proving that the sum of the reciprocals of the positive integers is infinite. Remember that to show divergence means showing that by adding sufficiently many terms, we can get the sum to exceed any bound given in advance. This is what we shall do for the harmonic series. First we group the positive integers into finite sets as follows: $\{1\}$, $\{2,3\}$, $\{4,5,6,7\}$, $\{8,9,10,11,12,13,14,15\}$, ... Observe that each set starts with a power of 2 and goes up to the number just short of the next higher power of 2. We now show the sum of the reciprocals of the numbers in each set exceeds $1/2$. For example, take the set $\{4,5,6,7\}$. Since 8 exceeds each number in the set, and there are 4 numbers in the set, we have:

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}>4\times\frac{1}{8}=\frac{1}{2}.$$

Similarly, take the set $\{8,9,10,11,12,13,14,15\}$. Since 16 exceeds each number in the set, and there are 8 numbers in the set, we have:

$$\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}>8\times\frac{1}{16}=\frac{1}{2}.$$

In the same way we show that for each set, the sum of the reciprocals exceeds $1/2$. So if we want the sum to, say, exceed 10, it suffices if we include 20 of these sets. Since the 20^{th} set has the integers from 2^{19} till $2^{20}-1=1048575$, this means that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{1048575}>10.$$

Hence the sum can exceed any bound given in advance. This means that the series diverges, as claimed. See reference [3] for more such proofs.

References

1 G H Hardy & E M Wright, *An Introduction to the Theory of Numbers*, Oxford (Fourth edition)

2 I Niven, H S Zuckerman & H Montgomery, *An Introduction to the Theory of Numbers*, John Wiley (Fifth edition)

3 http://en.wikipedia.org/wiki/Harmonic_series_(mathematics)

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