feature

Algebracadabra

Magic Squares . . .

. . . with prime entries

Magic squares have been a subject of fascination for centuries. Probably it is the elegance and the simplicity in the subject that attract people. Here we explore the question of how to construct magic squares composed solely of prime numbers.

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A magic square is an array of numbers in an $n \times n$ square grid arranged in such a way that the sum of the numbers in each row, each column and each of the two main diagonals is the same number (called the 'Magic Sum' of that square).

There are various ways of classifying magic squares; one is by the number of rows and columns. Thus we have *Odd-order Magic Squares*, with an odd number of rows, and *Even-order Magic Squares*, with an even number of rows. Even-order Magic Squares can be further classified into *Singly-even Magic Squares*, for which the number of rows is even but not a multiple of 4, and *Doubly-even Magic Squares*, for which the number of rows is a multiple of 4.

Constructing 3 × 3 Magic Squares

The numbers in any 3 × 3 magic square will always be formed using three Arithmetic Progressions (APs). Below is a 3 × 3 magic square where three APs have been placed in a particular pattern.

42	4	32
16	26	36
20	48	10

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Observe the following:

- The square is made up of three APs: (4, 10, 16), (20, 26, 32) and (36, 42, 48).
- Each row of the square contains one term of each of the APs. So also each column.
- The three APs have the same common difference (in this case 6).
- The first terms of the APs themselves form an AP, namely: (4, 20, 36).

To construct a 3 × 3 magic square, one has only to remember the above rules. This gives us the liberty to create 3 × 3 magic squares with arbitrary numbers of our choice provided they form APs related as above.

A thought occurs to us at this stage: *Can we construct magic squares composed of only prime numbers?*

Constructing 3 × 3 magic squares using only prime numbers

Constructing a 3 × 3 magic square using only prime numbers (we shall refer to such a square as a *Prime Magic Square*) is possible but challenging because we first need to list sufficiently many APs composed of primes, and they have to be interconnected in the right way. Shown below is an example of a prime magic square.

101	29	83
53	71	89
59	113	41

In going about this task, the following observation may be kept in mind: *If three prime numbers exceeding 3 form an AP, then the common difference of the AP is necessarily a multiple of* 6. The reader is invited to supply a proof of the statement. Using this observation simplifies the search for prime APs.

But how often do we come across prime APs with four or more terms? This is where constructing a 4 × 4 prime magic square becomes challenging, because four-term prime APs are a less frequent sight.

Constructing a 4 × 4 magic square

There are various methods for constructing 4×4 magic squares. Shown below is one of the methods used for positioning the numbers within it. The APs used here are:

AP(#1) = (1, 2, 3, 4), AP(#2) = (5 ,6, 7, 8),

AP(#3) = (9, 10, 11, 12), AP(#4) = (13, 14, 15, 16).

Here is how we arrange the different APs. We start with AP #1:

	1		
			2
		3	
4			

We may assign a symbol to fix or recall the above pattern: 2-4-3-1. Thus, the first term of the AP is placed in cell #2 of the first row, the second term of the AP is placed in cell #4 of the second row, the third term of the AP is placed in cell #3 of the third row, and the fourth term of the AP is placed in cell #1 of the fourth row.

Note the appearance of the L-shaped movement which reminds us of the knight move in chess. This is a theme which often crops up in the construction of magic squares.

Next we arrange the terms of AP #2:

	1		7
	8		2
5		3	
4		6	

This arrangement may be denoted by the symbol 4-2-1-3.

Following this step, we go on to arrange the terms of AP #3:

	1	12	7
11	8		2
5	10	3	
4		6	9

This arrangement may be denoted by the symbol 3-1-2-4. Here the term 9 (the first term in AP #3) is placed in the cell diagonally alternate to 8. (By 'diagonally alternate' we mean the cell which is 2 units to the right and 2 units below the starting cell. That is, the cell diagonally alternate to (i, j) is $(i+2, j+2)$, where addition is done modulo 4.) The second term in the AP is 10 which is placed in the cell diagonally alternate to 7, and it goes in this pattern for the remaining terms and for the final AP, the pattern being 1-3-4-2:

14	1	12	7
11	8	13	2
5	10	3	16
4	15	6	9

The final array is a magic square of order 4, with magic sum 34. The positions of numbers 1 to 16 define one way of positioning the numbers in the grid to get a magic square. (There are of course many other ways of doing this.) We can now take any four four-term APs and place them in these positions to get a 4 × 4 magic square.

Constructing a 4 × 4 prime magic square

Using the following selection of four-term APs composed of prime numbers,

AP(#1) = (11, 17, 23, 29),
AP(#2) = (41, 47, 53, 59),

AP(#3) = (61, 67, 73, 79),
AP(#4) = (251, 257, 263, 269),

we obtain the following 4 × 4 magic square with a magic sum of 400:

257	11	79	53
73	59	251	17
41	67	23	269
29	263	47	61

By selecting different sets of primes, we may generate infinitely more such magic squares. Try it out on your own!

Above we have a beautiful example of a 4 × 4 Prime Magic Square. But the square has an interesting feature which makes it different from the earlier 3 × 3 square: the first terms in the four APs (11, 41, 61, 251) do not themselves form an AP, yet the APs result in a Magic Square. This makes matters simpler for us: if we wish to construct a 4 × 4 Prime Magic Square, we only need four prime APs with the same common difference; their first terms need not form an AP. (In passing, we remark that there are various other combinations for constructing a 4 × 4 magic square where one starts from other cells and takes a knight move in a different direction. There are also methods of constructing such squares without taking the knight move.)

Constructing a 4 × 4 prime magic square without APs

Is it possible to construct a Prime Magic Square without using prime APs? Well, we will never know until we try, will we? Here's what we find when we make the attempt. We use the following four sequences of primes (note that they do not form APs but possess a definite structure):

Sequence(#1) = (29, 31, 59, 61),
Sequence(#2) = (71, 73, 101, 103),
Sequence(#3) = (149, 151, 179, 181),
Sequence(#4) = (197, 199, 227, 229).

Here we have two pairs of twin primes in every sequence such that the differences between the second and third terms in all the four sequences are the same. (Note: Twin primes are a pair of prime numbers that differ by 2. So they are a pair of consecutive odd numbers, both of which are prime.) Once again, the four sequences have to be similar in nature with regard to the common difference, but the first terms of the four sequences need not form an AP.

Using these we form the following 4 × 4 Prime Magic Square. The numbers in the four sequences are inserted into the cells according the same pattern used earlier: 2-4-3-1 for Sequence #1, 4-2-1-3 for Sequence #2, and so on. Here is what we get when we do this for all the four sequences:

199	29	181	101
179	103	197	31
71	151	59	229
61	227	73	149

This line of thinking opens up the possibility of exploring more types of 4 × 4 Prime Magic Squares, because the restriction of APs has been removed, as also the restriction that the first terms of the sequences should be in an AP/Sequence. A layman would be happy going this far. But for the mathematically inclined person, there arises the following question: *What is the logic behind Magic Squares?* Here we encounter the fascinating algebra of Magic Squares.

We select the four sequences according to the following pattern (note their structure):

Sequence(#1) = $(a, a+x, a+y, a+z)$,
Sequence(#2) = $(b, b+x, b+y, b+z)$,
Sequence(#3) = $(c, c+x, c+y, c+z)$,
Sequence(#4) = $(d, d+x, d+y, d+z)$.

Any four such sequences can be placed in the grid according to the pattern described earlier, and it will result in a Magic Square whose magic sum is $a+b+c+d+x+y+z$. Here is the result:

$d+x$	a	$c+z$	$b+y$
$c+y$	$b+z$	d	$a+x$
b	$c+x$	$a+y$	$d+z$
$a+z$	$d+y$	$b+x$	c

Closing remarks. The topic of Magic Squares can be further explored. What has been seen till now is just a tip of the iceberg. It is a good topic for exploration not just because of its recreational aspect but also because there are many areas of application. The subject indeed has Magic in it, for it manages to attract both young and old, student and teacher, layman and mathematician.

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