Solving a Famous Problem

The Chakravāla Method

Zeroing in on a Solution

feature

Let **n** *be a non-square positive integer. The problem of finding positive integers* $\mathbf{x}, \mathbf{y}$ *such that* $\mathbf{x^2 - ny^2 = 1}$ *is one of enormous interest to number theorists. This equation is commonly known today as 'Pell's equation', so-named after the English scholar John Pell (1611–1685). The name was given by Leonhard Euler (1707–1783), but we know now that this was done on an erroneous supposition. Before Pell and Euler, the equation had been explored by Pierre de Fermat. And much before Fermat — a whole millennium earlier — the same equation had been studied in great detail by the Indian mathematician Brahmagupta (598–670). The equation was referred to by Brahmagupta as the* ***Varga Prakriti****, or the "equation of the multiplied square". Some centuries later came Jayadeva (950–1000) who delved deeper into the problem and offered a more general procedure for its solution. Bhāskarachārya II (1114–1185) refined this work and gave a full account of the algorithm in his important work, Bījaganitam. He gave the name* ***Chakravāla*** *to the algorithm; the name reflects the cyclic or iterative nature of the procedure, for 'Chakra' means 'wheel'. Narayana Pandit (1340–1400) added further to this work. The above equation could therefore be called the Brahmagupta-Jayadeva-Bhāskarā equation, but we shall call it simply the Brahmagupta equation. In this note we explain the working of the Chakravāla algorithm to solve this equation. However, we do not give a proof that the algorithm works. For that we shall only refer you to published sources.*

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Introduction. Perhaps you have come across the 'house-number puzzle':

> *On a street with houses numbered* 1, 2, 3, ..., *A where A is a three-digit number, I live in a house, numbered B, such that the sums of the house numbers on the two sides of my house are equal. Find A and B.*

We can ask, more generally: "Find possible values of A and B" (without the condition that A is a three-digit number). The condition tells us that the sums $1 + 2 + \cdots + (B - 1)$ and

$(B+1)+(B+2)+\cdots+A$ are equal. Hence:

$$\frac{(B-1)B}{2}=\frac{A(A+1)}{2}-\frac{B(B+1)}{2}, \quad \therefore B^2=\frac{A(A+1)}{2}.$$

From the last relation we get: $8B^2=4A^2+4A$, hence $8B^2+1=(2A+1)^2$. Let $x=2A+1$ and $y=2B$. Then we have:

$$x^2-2y^2=1.$$

Thus the house-number puzzle has given rise to a Brahmagupta equation with $n=2$. This is one of several different ways in which we arrive at such equations.

As we have a single equation in two variables, there could be more than one solution. (There could also be no solutions.) Is there a systematic way of finding the solutions? This question was considered by Brahmagupta. He solved the problem for some individual values of n, e.g., $n=83$ and $n=92$ and went on to say: "A person who is able to solve these problems within a year is truly a mathematician!" A year? Brahmagupta clearly had a high regard for the problem!

Three identities

We start by some identities which were used with great effectiveness by the main actors in this story.

Diophantus-Brahmagupta-Fibonacci identity. The following two identities go back all the way to Diophantus of Alexandria (they are found in his book *Arithmetica*):

$$\begin{cases} (a^2+b^2)(c^2+d^2) &= (ad-bc)^2+(ac+bd)^2, \\ (a^2+b^2)(c^2+d^2) &= (ad+bc)^2+(ac-bd)^2. \end{cases} \tag{1}$$

The identities may be verified by expanding all the terms. They tell us that *the product of two integers, each a sum of two squares, is itself a sum of two squares, and in two different ways*. For example, consider the two integers $10=3^2+1^2$ and $41=5^2+4^2$; both are sums of two squares. Their product $10\times 41=410$ may be written in two such ways:

$$410=(15+4)^2+(12-5)^2=19^2+7^2,$$
$$410=(15-4)^2+(12+5)^2=11^2+17^2.$$

Today we describe this as a *closure result*. We say: *The set of integers expressible as a sum of two squares is closed under multiplication*.

How the early mathematicians hit upon this double identity is not clear, but it is easy for us today to reconstruct it and also to remember it by invoking the following result about complex numbers: *The magnitude of the product of two complex numbers is equal to the product of their magnitudes*. For, consider the complex numbers $a+bi$ and $c+di$. Here a,b,c,d are real numbers, and $i=\sqrt{-1}$. Now, observing that

$$|a+bi|^2=a^2+b^2=|a-bi|^2,$$
$$|c+di|^2=c^2+d^2=|c-di|^2,$$
$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i,$$
$$(a-bi)(c+di)=(ac+bd)+(ad-bc)i,$$

we see how the Diophantus-Brahmagupta-Fibonacci identity is equivalent to the statement "magnitude of product equals product of magnitudes" for complex numbers.

Brahmagupta's identity. Somewhat more general are the following identities which were first discovered by Brahmagupta and used extensively by him:

$$\begin{cases} \left(a^2 - nb^2\right)\left(c^2 - nd^2\right) &= (ac + nbd)^2 - n(ad + bc)^2, \\ \left(a^2 - nb^2\right)\left(c^2 - nd^2\right) &= (ac - nbd)^2 - n(ad - bc)^2. \end{cases} \tag{2}$$

Thus: *For each fixed integer n, the set of integers expressible as $x^2 - ny^2$ is closed under multiplication*. Example: Let $n = 2$. The integers $3^2 - 2 \times 1^2 = 7$ and $5^2 - 2 \times 2^2 = 17$ are both of the form $x^2 - 2y^2$, and so is their product $7 \times 17 = 119 = 19^2 - 2 \times 11^2$.

The case $n = -1$ corresponds to the Diophantus-Brahmagupta-Fibonacci identity.

As earlier, it is easy to verify these relations by expanding all the terms. And it is easy to reconstruct them by studying numbers of the form $a + b\sqrt{n}$, as can be seen by multiplying together two numbers of this form. For we have:

$$\left(a + b\sqrt{n}\right) \cdot \left(c + d\sqrt{n}\right) = (ac + nbd) + (ad + bc)\sqrt{n}. \tag{3}$$

Brahmagupta used (2) in his approach to the equation $x^2 - ny^2 = 1$. Indeed, he used it as a sort of 'composition law' which has a very modern look about it, for it looks like it is taken straight out of a book on modern algebra! In using such an approach, Brahmagupta was ahead of his time by a whole millennium.

There is another way of expressing the identity which may bring home its significance more strongly. *Suppose that $(x, y) = (a, b)$ is a solution of the equation $x^2 - ny^2 = k_1$, and $(x, y) = (c, d)$ is a solution of the equation $x^2 - ny^2 = k_2$. Then $(x, y) = (ac + nbd, ad + bc)$ is a solution of the equation $x^2 - ny^2 = k_1 k_2$.*

Remark. The appearance of the identical identity in the works of Fibonacci (1170–1250) and Brahmagupta may seem an amazing coincidence but it is not a mystery. Brahmagupta wrote an extremely important work, the *Brahma-Sphuta-Siddhantha*, which reached the centre of the Arab world (Baghdad) in the eighth century and was translated into Arabic, at the instance of the Caliph. Three centuries later this work was translated into Latin in Spain, and it is this doubly translated work that came into the hands of Fibonacci, who was then a trader in Italy.

Bhāskarā's lemma. The third identity in our list is the following, which is an auxiliary result needed by the Chakravāla algorithm:

$$\text{If } a^2 - nb^2 = k, \text{ then } \left(\frac{ma + nb}{k}\right)^2 - n\left(\frac{a + bm}{k}\right)^2 = \frac{m^2 - n}{k}. \tag{4}$$

The verification is straightforward:

$$\begin{aligned} \left(\frac{ma + nb}{k}\right)^2 - n\left(\frac{a + bm}{k}\right)^2 &= \frac{m^2a^2 + n^2b^2 + 2mnab - na^2 - nm^2b^2 - 2mnab}{k^2} \\ &= \frac{m^2a^2 + n^2b^2 - na^2 - nm^2b^2}{k^2} \\ &= \frac{(m^2 - n) \cdot (a^2 - nb^2)}{k^2} = \frac{m^2 - n}{k}. \end{aligned}$$

The Chakravāla

In the description given below, n is a fixed non-square positive integer. For convenience we use the symbol or triple $[x, y, k]$ to indicate that the numbers x, y, k satisfy the relation

$$x^2 - ny^2 = k. \tag{5}$$

Observe that n does not appear in the triple. This is so because n is treated as a fixed integer in these relations.

The Brahmagupta identity (2) shows that from two triples $[a, b, k_1]$ and $[c, d, k_2]$ we can produce a third triple $[u, v, k_1 k_2]$, with

$$u = ac + nbd, \qquad v = ad + bc. \tag{6}$$

Example: Take $n = 10$. We have the triples $[7, 2, 9]$ and $[11, 3, 31]$ which correspond to the relations

$$7^2 - 10 \times 2^2 = 9, \qquad 11^2 - 10 \times 3^2 = 31.$$

We seek a triple of the form $[u, v, 9 \times 31] = [u, v, 279]$. We get: $u = 77 + 10 \times 6 = 137$ and $v = 21 + 22 = 43$. Check:

$$137^2 - 10 \times 43^2 = 18769 - 10 \times 1849 = 279.$$

The composition law

$$[a, b, k_1] \star [c, d, k_2] := [ac + nbd, ad + bc, k_1 k_2] \tag{7}$$

was given the name *Samasa bhāvanā* by Brahmagupta (or just *Bhavana* for short).

Deriving Bhāskarā's lemma from the composition law. Bhāskarā's lemma stated above may look mysterious and unmotivated; but it looks more meaningful when we invoke the composition law. Say we have a triple $[a, b, k]$; that is, we have $a^2 - nb^2 = k$. For any integer m, we also have the 'trivial triple' $[m, 1, m^2 - n]$ whose second coordinate is 1; it corresponds to the trivial statement $(m)^2 - n \cdot 1^2 = m^2 - n$. Now let us compose these two triples using the Bhavana. We get:

$$[a, b, k] \star [m, 1, m^2 - n] = [ma + nb, a + bm, k(m^2 - n)]. \tag{8}$$

In other words we have: if $a^2 - nb^2 = k$, then

$$(ma + nb)^2 - n(a + bm)^2 = k(m^2 - n). \tag{9}$$

Dividing through by k^2 we get Bhāskarā's lemma:

$$\left(\frac{ma + nb}{k}\right)^2 - n\left(\frac{a + bm}{k}\right)^2 = \frac{m^2 - n}{k}. \tag{10}$$

Heuristic motivation for the Chakravāla. We seek to solve the equation $x^2 - ny^2 = 1$; that is, we seek a triple $[x, y, 1]$ whose third coordinate is 1, which is the smallest it can possibly be (it can never be 0, because n is non-square, implying that $\sqrt{n}$ is irrational). Now when we repeatedly apply the composition law, the number in the third coordinate steadily increases, for it is subject to integer multiplication, and this continues unless we effect a division at some stage. For this it must happen that $ma + nb$, $a + bm$ and $m^2 - n$ are all divisible by k. This is how Bhāskarā's lemma steps in, and this simple-minded reasoning is the heuristic logic behind the Chakravāla. Here, then, is how the algorithm due to Jayadeva, Bhāskarachārya II and Narayana Pandit works.

Steps of the Chakravāla algorithm.
Step 0: Start with a triple $[a, b, k]$ in which a and b are coprime.

Step 1: Look for values of m such that $a + bm$ is divisible by k. These values will form an arithmetic progression (AP) with common difference $|k|$. From this AP, select that value of m which makes $|m^2 - n|$ the least.

Step 2: Let a', b', k' be computed thus:

$$a' = \frac{ma + nb}{|k|}, \qquad b' = \frac{a + bm}{|k|}, \qquad k' = \frac{m^2 - n}{k}. \tag{11}$$

Replace $[a, b, k]$ by $[a', b', k']$. (That is, $[a, b, k] \longleftarrow [a', b', k']$, to use "computer grammar".) If the third coordinate of the new triple is 1, then this is the triple we seek; end of story. Else, go back to Step 1 and start a fresh cycle of computations.

The ‘cycle’ tells us why Bhāskarā called this the Chakravāla. We show the working of the algorithm using a few examples.

Example 1. Let $n = 10$. Let us start with the trivial triple $[4, 1, 6]$, that is, with the relation $4^2 - 10 \times 1^2 = 6$, and see where it leads us.

Step 0: $[a, b, k] = [4, 1, 6]$.

Step 1: We want $a + bm$ to be divisible by k, i.e., $4 + m$ to be divisible by 6. Hence $m \in \{2, 8, 14, 20, \ldots\}$. The value of m for which $m^2 - 10$ is least is $m = 2$.

Step 2: $a' = \dfrac{8 + 10}{|6|} = 3,\ b' = \dfrac{4 + 2}{|6|} = 1$ and $k' = \dfrac{4 - 10}{6} = -1$.

We have obtained the triple $[a, b, k] = [3, 1, -1]$. Since the third coordinate is not 1, we go back to Step 1.

Step 1: We want $3 + m$ to be divisible by -1. Any integer value will do. We also want $|m^2 - 10|$ to be as small as possible. So we choose $m = 3$.

Step 2: $a' = \dfrac{9 + 10}{|-1|} = 19,\ b' = \dfrac{3 + 3}{|-1|} = 6$ and $k' = \dfrac{9 - 10}{-1} = 1$.

We have obtained the triple $[a, b, k] = [19, 6, 1]$. Now the third coordinate is 1, so the computations are over, and we have obtained a solution to the problem:

$$19^2 - 10 \times 6^2 = 1.$$

Example 2. Let $n = 13$. We start with the trivial triple $[4, 1, 3]$, that is, with the relation $4^2 - 13 \times 1^2 = 3$.

Step 0: $[a, b, k] = [4, 1, 3]$.

Step 1: We want $a + bm$ to be divisible by k, i.e., $4 + m$ to be divisible by 3. Hence $m \in \{2, 5, 8, 11, \ldots\}$. The value of m for which $m^2 - 13$ is least is $m = 2$.

Step 2: $a' = \dfrac{8 + 13}{|3|} = 7,\ b' = \dfrac{4 + 2}{|3|} = 2$ and $k' = \dfrac{4 - 13}{3} = -3$.

We have obtained the triple $[a, b, k] = [7, 2, -3]$. As the third coordinate is not 1, we go back to Step 1.

Step 1: We want $7 + 2m$ to be divisible by -3. Hence $m \in \{1, 4, 7, 10, \ldots\}$. The value of m for which $|m^2 - 13|$ is least is $m = 4$.

Step 2: $a' = \dfrac{28 + 26}{|-3|} = 18,\ b' = \dfrac{7 + 8}{|-3|} = 5$ and $k' = \dfrac{16 - 13}{-3} = -1$.

We have obtained the triple $[a, b, k] = [18, 5, -1]$. We go back to Step 1.

Step 1: We want $18 + 5m$ to be divisible by -1. Any integer value will do. We want m such that $|m^2 - 13|$ is least. So we choose $m = 4$.

Step 2: $a' = \dfrac{72 + 65}{|-1|} = 137,\ b' = \dfrac{18 + 20}{|-1|} = 38$ and $k' = \dfrac{16 - 13}{-1} = -3$.

We have obtained the triple $[a, b, k] = [137, 38, -3]$. Back to Step 1.

Step 1: We want $137 + 38m$ to be divisible by -3. Hence $m \in \{2, 5, 8, 11, \ldots\}$. The value of m for which $|m^2 - 13|$ is least is $m = 2$.

Step 2: $a' = \dfrac{274 + 494}{|-3|} = 256,\ b' = \dfrac{137 + 76}{|-3|} = 71$ and $k' = \dfrac{4 - 13}{-3} = 3$.

We have obtained the triple $[a, b, k] = [256, 71, 3]$. Back to Step 1.

Step 1: We want $256 + 71m$ to be divisible by 3. Hence $m \in \{1, 4, 7, 10, \ldots\}$. The value of m for which $|m^2 - 13|$ is least is $m = 4$.

Step 2: $a' = \dfrac{1024 + 923}{|3|} = 649$, $b' = \dfrac{256 + 284}{|3|} = 180$ and $k' = \dfrac{16 - 13}{3} = 1$.

We have obtained the triple $[649, 180, 1]$. Now the third coordinate is 1 so the computations are over. Here is our solution:

$$649^2 - 13 \times 180^2 = 1.$$

Remark. We have described the working of the Chakravāla, but we have not attempted to show that it will always yield an answer. For this, we must show that we will reach a triple $[u, v, 1]$ no matter with which triple $[a, b, k]$ we start the computation; of course, we do need GCD $(a, b) = 1$. Interested readers may refer to [1] or [3] for proof that the Chakravāla algorithm works.

Had the ancients considered such a question? Did Jayadeva, Bhāskarā and Narayana Pandit *know* that the Chakravāla algorithm would invariably give a solution? We do not know.

In closing we note that there are some steps that serve as shortcuts. Brahmagupta found that if we ever reach a triple $[a, b, k]$ with $k \in \{\pm 4, \pm 2, -1\}$, then we can reach the desired end (with third coordinate equal to 1) in a single step, thus shortening the computations greatly. We do not study these shortcuts here. Please refer to [3] for details.

Exercises. Using the Chakravāla, find some positive integer solutions to the following equations. Use any convenient starting triple of your own choice in each case.

(1) $x^2 - 8y^2 = 1$

(2) $x^2 - 11y^2 = 1$

(3) $x^2 - 15y^2 = 1$

(4) $x^2 - 17y^2 = 1$

(5) $x^2 - 19y^2 = 1$

(6) $x^2 - 41y^2 = 1$

Remark. Using the Chakravāla, Bhāskarā found the following positive integer solution to the equation $x^2 - 61y^2 = 1$ (it is the *least* such!): $x = 1766319049$, $y = 226153980$.

References

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