Bījagaṇita of Bhāskarācārya

feature

What would have been Fermat's astonishment if some missionary, just back from India, had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier!

Amartya Kumar Dutta

The above sentence occurs in the book "Number Theory: An approach through history" (p 81–82) by André Weil (1906–98), one of the giants of 20th century mathematics. The mathematician Pierre de Fermat (1601–65) is regarded as the father of modern number theory. The "problem" referred to by Weil has a grand history. It was posed by Fermat in 1657 as part of his efforts to kindle the interest of contemporary mathematicians in the abstract science of numbers. The problem was to find all *integers* (or whole numbers) x, y which satisfy the equation $Dx^2 + 1 = y^2$, where D is a fixed positive integer which is not a perfect square. The unexpected intricacy of the problem can be felt from the case $D = 61$: the smallest solution (in positive integers) of the equation $61x^2 + 1 = y^2$ is $x = 226153980$, $y = 1766319049$. In his challenge, Fermat had specifically highlighted this case ($D = 61$).

The above problem turned out to be of paramount importance in algebra and number theory. It fascinated some of the greatest mathematicians of modern Europe like Euler (1707–83) and Lagrange (1736–1813). Powerful theories and techniques emerged out of the researches centred around the equation.

Keywords: *History of mathematics, ancient India, Fermat, Brahmagupta, Chakravala, Bhaskaracharya, infinity, mathematical verse, metre, anustup*

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A thousand years before Fermat, the ancient Indian mathematician-astronomer Brahmagupta (628 CE) had investigated the same problem and came up with a brilliant composition law *bhāvanā* on the solution space of the more general equation $Dx^2 + m = y^2$. Brahmagupta's work anticipates several basic principles of modern number theory and abstract algebra. Using Brahmagupta's rule, subsequent Indian algebraists developed an astonishing algorithm called *cakravāla* which gives a complete solution to the problem. The algorithm was discovered by the 11th century — a work of 1073 CE quotes the algebraist Jayadeva's solution to the problem!

The algebra text **Bījagaṇita** (1150 CE) of **Bhāskarācārya** (b. 1114 CE) gives a brief but lucid description, in Sanskrit verses, of the *bhāvanā* law followed by the *cakravāla* algorithm for solving Fermat's equation $Dx^2 + 1 = y^2$. The method is illustrated by two difficult examples including the peculiar example $D = 61$. No wonder that glowing tributes were paid by European scholars after a translation of the *Bījagaṇita* was published. The German mathematician H. Hankel (1874) wrote about the *cakravāla* method:

> It is beyond all praise: it is certainly the finest thing achieved in the theory of numbers before Lagrange.

Bhāskarācārya's *Bījagaṇita* also discusses integer solutions to the linear equation $ax - by = c$, a problem with applications in astronomy and calendar-making. The problem had been solved by Āryabhaṭa (499 CE) by a method called *kuṭṭaka* (pulverisation) which involves a subtle idea resembling Fermat's celebrated principle of descent. There are various other interesting examples of problems involving integer solutions in *Bījagaṇita*. From the verses, one can get a glimpse of the thrill and delight that the ancient Indian algebraists felt in handling such difficult number-theoretic problems. Even today, not many high-school students (or even college students) in India are familiar with this important branch of mathematics.

Bījagaṇita also covers topics in basic algebra that are now familiar to high-school students: negative numbers and zero, variables (unknowns), surds, and the fundamental operations with them; solutions of simultaneous equations in several unknowns; and the solution of the quadratic equation by the method of "elimination of the middle term" (or "completing the square") — an idea with far-reaching consequences in mathematics. As in modern school-texts, interesting concrete examples are given to illustrate applications of the principles.

Bhāskarācārya took the bold step of introducing infinity in mathematics and defining rules of interactions with usual numbers: $\infty + x = \infty$ and $\infty - x = \infty$. The idea of adjunction of infinity has now been put on a firm footing in several branches of higher mathematics like analysis or the valuation theory in commutative algebra and number theory.

The verses in *Bījagaṇita* are in the *anuṣṭup* metre. Ancient Indians had the perception that the metrical form has greater durability, power, intensity and force than the unmetrical and invariably recorded all important knowledge in verse form. It could be exciting for a modern reader to watch how Bhāskarācārya moulds the Sanskrit language to present technical terms and hard results of mathematics in the verse format!

Touches of mythological allegories enhance the charm of Bhāskarācārya's *Bījagaṇita*. While discussing properties of the mathematical infinity, Bhāskarācārya draws a parallel with Lord Viṣṇu who is referred to as *Ananta* (endless, boundless, eternal, infinite) and *Acyuta* (firm, solid, imperishable, permanent):

> During *pralay* (Cosmic Dissolution), beings merge in the Lord and during *sṛṣti* (Creation), beings emerge out of Him; but the Lord Himself — the *Ananta*, the *Acyuta* — remains unaffected. Likewise, nothing happens to the number infinity when any (other) number enters (i.e., is added to) or leaves (i.e., is subtracted from) the infinity; it remains unchanged.

The use of a mystic metaphor to explain the mathematical principle $\infty \pm x = \infty$ reflects the vibrant culture of the bygone era. Perhaps the spiritual culture had prepared the Indian mind for, and probably suggested to it, the concept of the mathematical infinity (or zero!) with its curious properties.

Again, in order to emphasise the importance, power and profundity of algebra, Bhāskarācārya begins the treatise with an Invocation involving an interesting "pun" on the words *Sāṁkhyāḥ* (the Sāṁkhya philosophers as well as the experts in *saṁkhyā*, the science of numbers), *Satpuruṣa* (the Self-Existent Being as well as the wise mathematician), *bīja* (root/cause as well as algebra) and *vyakta* (the manifested universe as well as the revelation of an unknown quantity).Thus, through the opening verse, Bhāskarācārya venerates the Unmanifested — the Self-Existent Being of the Sāṁkhya philosophy — who is the originator of intelligence and the primal Cause of the known or manifested universe; and, through the very same words, Bhāskarācārya pays tribute to the wise mathematician who, using algebra, solves a problem (i.e., reveals or manifests an unknown quantity)!

The importance of algebra is reiterated at the end of *Bījagaṇita*. Bhāskarācārya remarks that algebra is the essence of all mathematics, is full of virtues and free from defects; and that cultivation of algebra will sharpen the intellect of children. He concludes with the exhortation "*paṭha paṭha*" (Learn it, learn it) for the development of intelligence.

In this connection, I may mention here that one of our greatest contemporary mathematicians Shreeram S. Abhyankar (b. 1930) acknowledges the influence of Bhāskarācārya during his formative years. Abhyankar fondly recalls how his father (S.K. Abhyankar) used to teach him mathematics by reciting to him lines from Bhāskarācārya's text *Līlāvatī* and how he used to memorise them when he was around ten years of age.

Bhāskarācārya's *Bījagaṇita* not only makes us aware of the great advancements made by ancient Indian algebraists, it also gives us a feel for the charming atmosphere in which mathematical research and discourse — at both basic and advanced levels — used to take place in ancient times. While a study of *Bījagaṇita* will be enriching and inspiring for all cultured students of mathematics, a careful analysis of the treatise will also provide valuable insights to historians and scholars in general.

End-notes

- The quote (1874) by Hankel appeared in *Zur Geschichte der Mathematik in Alterthum und Mittelalter* (Leipzig, 1874), p 202.
- The remark by S. S. Abhyankar occurs on page 135 of his survey article "Resolution of Singularities and Modular Galois Theory", Bulletin of the American Mathematical Society, Vol. 38(2) (2001). The article is reproduced in the volume "Connected at Infinity: A Selection of Mathematics by Indians" ed R. Bhatia, pub Hindustan Book Agency (TRIM 25); the remarks occur in p 177.

End Notes for 'Bījagaṇita of Bhāskarācārya'

(1) The above article by Prof Amartya Kumar Dutta was published in the magazine *Prabuddha Bharata*, in the Sept 2007 issue (pages 545–546). It is reproduced here by kind permission of its editor, Swami Narasimhananda. For information on this publication, kindly refer to the website www.advaitaashrama.org.

(2) In connection with the sentence "In this connection, I may mention here that one of our greatest contemporary mathematicians Shreeram S. Abhyankar (b. 1930) ..." which appears in Prof Dutta's article, please note that this article was written in 2007. Unfortunately, Prof. Abhyankar passed away in November 2012.

(3) In the article there is a reference to the '*anuṣṭup* metre'. As the author has noted, it was the practice of ancient Indians to record all important knowledge in verse form. One of the metres they made use of frequently is *anuṣṭup*. For more information on this, please refer to https://en.wikipedia.org/wiki/Anustubh.

(4) In the same issue of *Prabuddha Bharata* (Sept 2007), there is an article on Indian mathematics by Prof. Kumar Murty (see http://www.advaitaashrama.org/Content/pb/2007/092007.pdf), which has a prescient paragraph on Prof Manjul Bhargava, the Indian-origin mathematician who was honoured with a Fields Medal at the International Congress of Mathematicians held in Seoul, South Korea, in August 2014: "The work of Bhargava, who is currently Professor of Mathematics at Princeton University, is deep, beautiful, and largely unexpected. It has many important ramifications and will likely form a theme of mathematical study at least for the coming decades."

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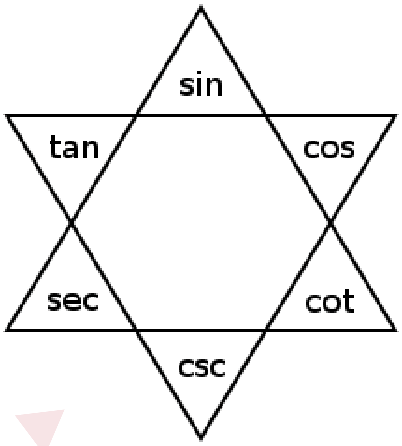
ALISON'S TRIANGLE

Here is a simple diagram which helps in remembering some trig identities.
It has been taken from the webpage
http://www.futilitycloset.com/2014/04/20/alisons-triangle/
It allows us to reconstruct all relations of the form a÷b=c or a×b=c, where a,b,c are the basic trigonometric functions of the same angle θ.

Here is the diagram :



Here's how it is used. Take any three functions arranged next to each other (e.g., sin, cos, cot). Then the product of the ends equals the middle.

For example, from "sin cos cot" we get: $\sin\theta \times \cot\theta = \cos\theta$.

Another way of stating this: the middle function divided by a function at the end equals the function at the other end.
For example, from "sec tan sin" we get : $\tan\theta \div \sin\theta = \sec\theta$.

If we commit this diagram to memory, we can reconstruct all possible such relations. On the webpage mentioned, the writer notes that he found the diagram in Michael Stueben's book Twenty Years Before the Blackboard (Spectrum, 1998).